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Gauge Coupling Constants as Residues of Spacetime Representations

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The gauge coupling constants in the electroweak standard model can be written as mass ratios, e.g. the coupling constant for isospin interactions $g_2^2 = 2\frac{m_W^2}{m^2} \sim 2(\frac{80}{169})^2 \sim \frac{1}{2.3}$ with the mass of the charged weak boson and the mass parameter characterizing the ground state degeneracy. A theory is given which relates the two masses in such a ratio to invariants which characterize the representations of a noncompact nonabelian group with real rank 2. The two noncompact abelian subgroups are operations for time and for a hyperbolic position space in a model for spacetime, homogeneous under dilation and Lorentz group action. The representations of the spacetime model embed the bound state representations of hyperbolic position space as seen in the nonrelativistic hydrogen atom. Interactions like Coulomb or Yukawa interactions are described by Lie algebra representation coefficients. A quantitative determination of the ratio of the invariants for position- and time-related operations, determined by the spacetime representation, gives the right order of magnitude for the gauge coupling constants.

KEY WORDS: gauge coupling constants; representation invariants; residual normalizations.

1. INTRODUCTORY REMARKS

In the following, the radical point of view is taken that all basically relevant physical properties, e.g. energies, momenta, masses, spin, helicity, charges and also coupling constants, can be understood as invariants and eigenvalues connected with the action of operations from real finite dimensional Lie groups. It helps in this way and will be shown in this paper that the basically relevant wave functions, like the hydrogen bound states or the on-shell part of the particle Feynman propagators, are representation matrix elements of operation groups and that the basic interactions, like Coulomb or Yukawa potentials or the off-shell part of Feynman propagators, are representation coefficients of the corresponding Lie algebras as tangent operations. Sometimes, there will arise quite a new and,

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perhaps, unfamiliar group- and operation-oriented language for well-known physical concepts.

Spacetime is an operational concept, physically interpretable by its representations. It will be modeled by a homogeneous space of a group. Therefore, in the end, an understanding of spacetime is reduced to an understanding of the representations of the underlying group and—for its interactions and its particles—of the tangent translations, i.e. of the corresponding Lie algebra representations.

Masses of particles and coupling constants, especially for mass zero particles, will be related to invariants of translation representations (Wigner, 1939) and the normalization of such representations as arising from product representations of an underlying nonabelian group. The formulation in terms of 'residual representations' (Saller, 2001a) leads to an interpretation of invariants and normalizations as complex poles and their residues.

The numerical values of gauge coupling constants seem not to be rational numbers as shortly sketched later (Section 3) in the context of the standard model for the electroweak and electrostrong interactions (Weinberg, 1967). If the values are from a continuous spectrum, the related operations have to include a noncompact group, since all properties from compact operations are given by rational numbers (Fulton, 1991), e.g. (hyper)charge numbers, angular momenta (spin), color dimensions, etc. The nontrivial representation structure of a Lie group, which is both noncompact and nonabelian, e.g. of the Lorentz group, is—for representations with a probability inducing Hilbert product—infinite dimensional.

In a certain sense, Sections 4–6 serve as an introduction to the theory of noncompact group representations as applied for an understanding of the gauge coupling constants as residues of spacetime representations. They contain a formulation of quantum mechanical free scattering and bound states and of free particles in quantum field theory to exemplify—without interaction—the use of translation representations and—for interaction—the use of nonabelian hyperbolic representation structures.

After this preparation, the spacetime representation theory for relativistic particles and interactions can be seen as a generalization and an embedding of these structures.

2. BASIC REPRESENTATION THEORY AND NOTATIONS

The level of the mathematical tools to treat noncompact nonabelian Lie groups is not undergraduate: It is difficult from the conceptual point of view (Knapp, 1986) and looks complicated in the explicit formulation. In this section, some basic representation theory (Folland, 1995; Kirillov, 1976; Knapp, 1986) is shortly summarized, which will be used later in the paper in many physical examples.

There will be considered representations of real Lie groups G on complex vector spaces with action $|a\rangle \longmapsto g \bullet |a\rangle$ for $g \in G$. If not stated otherwise, all these representations are meant as Hilbert representations, i.e. acting upon a Hilbert space with probability inducing invariant scalar product $\langle a|b\rangle$. Faithful representations of noncompact Lie groups are infinite dimensional.

For a locally compact group G, all representations are direct sums of cyclic representations and direct integrals of irreducible ones. A cyclic representation space is the closed complex span of the group orbit of a vector $G \bullet |c\rangle$—called a cyclic vector. Irreducible representations are cyclic—not necessarily vice versa.

For a compact group U, all representations are direct sums of irreducible finite-dimensional representations—there, the additional concept 'cyclic' is not important. With the Plancherel measure a counting measure the direct integrals are direct sums.

All representations of a locally compact group involve complex group functions acted upon with the both sided regular group representation. Their values are representation matrix elements (coefficients) $G \ni g \longmapsto d(g) = \langle a|g \bullet |b\rangle$. The dual of the algebra with the continuous compactly supported functions $\mathcal{C}_c(G)$ is the convolution algebra with the Radon measures $\mathcal{M}(G)$, which are generalized functions (distributions) with Haar measure basis. The Dirac measures embed the group. The Lebesgue spaces $L^p(G)$, $1 \leq p \leq \infty$, contain the self-dual Hilbert space $L^2(G)$ with the square integrable, the convolution algebra $L^1(G)$ with the absolute integrable and its dual $L^\infty(G)$ with the essentially bounded function classes. $L^1(G)$ can be considered to constitute a both sided ideal in the Radon distributions. For a compact group, the algebra $L^1(U)$ contains all Lebesgue spaces, $L^p(U) \supseteq L^q(U)$ for $p \leq q$. The functions, basically relevant for free and interacting physical structures in spacetime, will be given explicitly later.

The Hilbert spaces with representations of a locally compact group can be constructed with equivalence classes of functions from $L^1(G)$. They are characterized by their scalar product: There is a bijection between functions, which induce a scalar product on $L^1(G)$

$$\langle f|f'\rangle_d = f_{G\times G}\, dg_1\, dg_2 \overline{f(g_1)} d\left(g_1^{-1}g_2\right) f'(g_2)$$

the so-called positive-type functions (Gel'fand and Raikov, 1942) $d \in L^\infty(G)_+$ from the dual of the Lebesgue convolution group algebra, and equivalence classes of cyclic G-representations. Not all Hilbert spaces with the action of a noncompact locally compact group have to be constituted by square integrable function classes.

The positive-type functions constitute a cone. Their conjugation goes with the group inversion $d(g) = \overline{d(g^{-1})}$ and they are bounded by the value at the neutral element $|d(g)| \leq d(e)$. The conjugated partner of a positive-type functions for a cyclic representation characterizes the dual representation (Bourbaki, 1989)

$d \leftrightarrow \bar{d}$ and a real positive-type function $d = \bar{d}$ a self-dual cyclic representation. Many explicit examples will be given later.

A positive-type function gives the group expectation values of a cyclic vector $g \longmapsto d(g) = \langle c|g \bullet |c\rangle$. If normalized at the group unit, it is called a state. With a normalized cyclic vector, group representation and quantum probability normalization are related to each other.

An extremal element in the convex set of states, i.e. not combinable with strictly positive numbers from other states, is called a pure state. A cyclic vector for a pure state is called a pure cyclic vector. There is a bijection between pure states and equivalence classes of irreducible G-representations.

Summarizing these concepts and their notations

positive-type function (cyclic) $d \in L^\infty(G)_+$ with cyclic vector $G \ni g \longmapsto d(g) = \langle c\|g \bullet \|c\rangle$
pure state (irreducible) with normalized pure cyclic vector $G \ni e \longmapsto d(e) = \langle \Omega\|\Omega\rangle = 1$

$$G \subseteq \mathcal{C}_c(G)' = \mathcal{M}(G) \supseteq L^1(G),\ L^1(G)' = L^\infty(G)$$

Group functions contain the functions of group classes G/H with a closed subgroup, i.e. functions on symmetric (homogeneous) G-spaces. The H intertwiners of G, valued in a H-representation space, are acted upon with induced G-representations (Mackey, 1951), in general not irreducible.

Some notation for symmetric spaces, relevant in the following: The spheres and hyperboloids parameterize the orientation manifolds of compact rotations $\mathbf{SO}(s)$ either in compact rotations $\mathbf{SO}(1+s)$ or in noncompact Lorentz transformations $\mathbf{SO}_0(1,s)$. They get symbols (instead of the elsewhere also used S^s and H^s), which look a bit similar to the one-dimensional circle Ω^1 and one branch hyperbola $\mathcal{Y}^1$

$$\begin{aligned}
\text{spheres:}\quad \Omega^s &\cong \mathbf{SO}(1+s)/\mathbf{SO}(s), \quad s = 1, 2, \ldots, \Omega^0 = \{\pm 1\} \\
\Omega^1 &\cong \mathbf{SO}(2) \cong \mathbf{U}(1), \qquad \Omega^3 \cong \mathbf{SU}(2) \\
\text{hyperboloids:}\quad \mathcal{Y}^s &\cong \mathbf{SO}_0(1,s)/\mathbf{SO}(s), \quad s = 1, 2, \ldots \\
\mathcal{Y}^1 &\cong \mathbf{SO}_0(1,1) \cong \mathbf{D}(1), \qquad \mathcal{Y}^3 \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2).
\end{aligned}$$

The three-dimensional hyperboloid $\mathcal{Y}^3$ is the orientation manifold of position spaces in Minkowski spacetime, it characterizes special relativity. In contrast to the hyperboloids, the spheres have a finite measure (area, volume) $|\Omega^s| = \frac{2\pi^{\frac{1+s}{2}}}{\Gamma(\frac{1+s}{2})}$. A semidirect affine group $G \vec{\times} \mathbb{R}^n$ has cosets $G \vec{\times} \mathbb{R}^n / G \cong \mathbb{R}^n$ with respect to the homogeneous group $G \subseteq \mathbf{GL}(\mathbb{R}^n)$. They can be parameterized by $\mathbb{R}^n$ as symmetric space, in general not as additive group.

The two abelian Lie groups, the compact one-torus $\mathbf{U}(1)$ and its noncompact cover, the additive group $\mathbb{R}$ with multiplicative notation $\mathbf{D}(1) = \exp \mathbb{R}$ (dilations), are the Cartan subgroup types. The abelian subgroups in semi-simple groups come in their self-dual representations, i.e. as abelian axial rotations $\mathbf{SO}(2)$ and as abelian Lorentz transformations $\mathbf{SO}_0(1, 1)$ respectively.

There is a big difference between orthogonal groups $\mathbf{SO}_0(t, s)$ for even dimensions $t + s = 2R$ and odd dimensions $t + s = 1 + 2R$ with rank $R = 1, 2, \ldots$ as seen, e.g., in the centers of their covering groups and the number of spinor representations. This difference can be illustrated also with the Lorentz group in four dimensions (Gel'fand and Neumark, 1957; Neumark, 1958) $\mathbf{SO}_0(1, 3) \sim \mathbf{SL}(\mathbb{C}^2)$ with one type of Cartan subgroups $\mathbf{SO}(\mathbb{C}^2) = \mathbf{SO}(2) \times \mathbf{SO}_0(1, 1)$ and the Lorentz group in three dimensions (Bargmann, 1947) $\mathbf{SO}_0(1, 2) \sim \mathbf{SL}(\mathbb{R}^2)$ with two Cartan subgroup types $\mathbf{SO}(2)$ and $\mathbf{SO}_0(1, 1)$. The theory given later is built for odd-dimensional position rotations $\mathbf{SO}(2R - 1)$, nontrivial for $R \geq 2$, as subgroups of orthochronous Lorentz groups $\mathbf{SO}_0(1, 2R - 1)$ acting on even-dimensional spacetimes with one-dimensional time. For an odd-dimensional spacetime with even-dimensional position, e.g. Bargman spacetime with $\mathbf{SO}_0(1, 2)\vec{\times}\mathbb{R}^3$ (Bargmann, 1947; Saller, 2004) replacing Minkowski spacetime with $\mathbf{SO}_0(1, 3)\vec{\times}\mathbb{R}^4$, the theory would look very different.

The cyclic groups $\mathbb{Z}_R \cong \mathbb{Z}/R\mathbb{Z}$ (rest classes $k \bmod R$ with additive notation) will be used also in a multiplicative notation $\mathbb{I}(R)$ (complex unit roots $z^R = 1$). The sign and step functions are representations of the reals in $\mathbb{I}(2) \cong \mathbb{Z}_2 \cong \Omega^0$ and will be denoted as follows

$$\mathbb{R} \ni x \longmapsto \epsilon(x) = \frac{x}{|x|} \in \{\pm 1\} \cong \mathbb{I}(2) \cong \Omega^0$$

$$\mathbb{R} \ni x \longmapsto \vartheta(\pm x) = \frac{1 \pm \epsilon(x)}{2} \in \{0, 1\} \cong \mathbb{Z}_2$$

The step functions give the characteristic functions for future $\mathbb{R}_+$ and past $\mathbb{R}_-$.

With a group G-representation and its complex functions there is also the representation of its Lie algebra, denoted as its logarithm $\log G$. The tangent structure of a noncompact nonabelian group and its homogeneous spaces G/H will be interpreted as interactions for the spaces G/H. The more abstract formulation of Lie algebra representations coefficients and the physical realization as interactions for position space and spacetime is taken up again in Section 10.

3. STANDARD MODEL OF GAUGE INTERACTIONS

For an experimental orientation, the order of magnitude of gauge coupling constants is given as used in the standard model of particle interactions.

Electrodynamics connect the translations $\mathbb{R}^4$ of Minkowski spacetime with internal $\mathbf{U}(1)$-transformations. The charge iQ implements the action of the Lie algebra $\log \mathbf{U}(1)$. Its position density leads to currents $\mathbf{J}_k$, e.g. for a quantum Dirac field $\{\bar{\Psi}, \Psi\}(\vec{x}) = \gamma^0 \delta(\vec{x})$ with integer $\mathbf{U}(1)$-winding number (charge number) z

$$\mathbf{U}(1): \qquad \Psi \longmapsto e^{iz\alpha}\Psi, \qquad \bar{\Psi} \longmapsto e^{-iz\alpha}\bar{\Psi}, \qquad z \in \mathbb{Z}$$

$$\text{for } \log\mathbf{U}(1): \quad \mathbf{J}_k = z\frac{[\Psi_{\gamma k}, \bar{\Psi}]}{2}, \qquad Q = \int d^3x \mathbf{J}_0(x), \qquad \begin{cases} [Q, \Psi] = z\Psi \\ [Q, \bar{\Psi}] = -z\bar{\Psi} \end{cases}$$

The $\mathbf{U}(1)$-gauge dynamics is characterized by the classical Lagrangian with the individual kinetic Lagrangians and the electromagnetic interaction

$$\mathbf{L}(\mathbf{A}) + \mathbf{L}(\Psi) - \mathbf{A}^k \mathbf{J}_k, \qquad \begin{cases} \mathbf{L}(\mathbf{A}) = \mathbf{F}_{kj}\frac{\partial^k \mathbf{A}^j - \partial^j \mathbf{A}^k}{2} + g^2 \frac{\mathbf{F}_{kj}\mathbf{F}^{kj}}{4} \\ \mathbf{L}(\Psi) = i\bar{\Psi}\partial^k \gamma_k \Psi + m\bar{\Psi}\Psi \end{cases}$$

The constant g^2 is the electromagnetic coupling constant, related to Sommerfeld's fine structure constant α_{S}, with the experimental value

$$\alpha_{\mathrm{S}} = \frac{g^2}{4\pi} \sim \frac{1}{137.036}, \qquad g^2 \sim \frac{1}{10.9}$$

The coupling constant is the normalization of the gauge field $\mathbf{A}$ and related to the residue at the mass zero pole in the Feynman propagator

$$\langle\{\mathbf{A}^k, \mathbf{A}^j\}(x) - \epsilon(x_0)[\mathbf{A}^k, \mathbf{A}^j](x)\rangle = \frac{i}{\pi}\int \frac{d^4q}{2\pi} \frac{-\eta^{kj} g^2}{q^2 + io} e^{iqx}$$

$$\langle[\bar{\Psi}, \Psi](x) - \epsilon(x_0)\{\bar{\Psi}, \Psi\}(x)\rangle = \frac{i}{\pi}\int \frac{d^4q}{2\pi} \frac{\gamma^k q_k + m}{q^2 + io - m^2} e^{iqx}$$

$$\underset{\text{on-shell}}{\delta(q^2 - m^2)} + \underset{\text{off-shell}}{\frac{i}{\pi}\frac{1}{q_{\mathrm{P}}^2 - m^2}} = \frac{i}{\pi}\frac{1}{q^2 + io - m^2}, \text{ P principal value}$$

All electromagnetic interactions are quantitatively determined with the value of the gauge coupling constant g^2 and the representation characteristic integer $\mathbf{U}(1)$-winding numbers $z \in \mathbb{Z}$.

The minimal standard model of the elementary interactions in Minkowski spacetime $\mathbb{R}^4$ is a theory of compatibly represented external Poincaré group and internal 'charge-like' operations. It embeds the electromagnetic interaction for an electron Dirac field (quantum electrodynamics) into the electroweak and electrostrong interactions of lepton and quark Weyl fields. The fields involved are acted upon with irreducible representations $[2L|2R]$ of the Lorentz (cover) group $\mathbf{SL}(\mathbb{C}^2$ and irreducible representations $[y]$, $[2T]$ and $[2C_1, 2C_2]$ of the hypercharge group $\mathbf{U}(1)$ (rational hypercharge number y), isospin group $\mathbf{SU}(2)$ (integer or half-integer isospin T) and color group $\mathbf{SU}(3)$ (characterized by two integers $2C_{1,2}$) as given

Table I. Fermion and Gauge Fields of the Standard Model

Field	Symbol	$\mathrm{SL}(\mathbb{C}^2)[2L\|2R]$	$\mathbf{U}(1)[y]$	$\mathbf{SU}(2)[2T]$	$\mathbf{SU}(3)[2C_1, 2C_2]$
Left lepton	$\mathbf{l}$	[1|0]	$-\frac{1}{2}$	[1]	[0,0]
Right lepton	$\mathbf{e}$	[0|1]	-1	[0]	[0,0]
Left quark	$\mathbf{q}$	[1|0]	$\frac{1}{6}$	[1]	[1,0]
Right up quark	$\mathbf{u}$	[0|1]	$\frac{2}{3}$	[0]	[1,0]
Right down quark	$\mathbf{d}$	[0|1]	$-\frac{1}{3}$	[0]	[1,0]
Hypercharge gauge	$\mathbf{A}_0$	[1|1]	0	[0]	[0,0]
Isospin gauge	$\vec{\mathbf{A}}$	[1|1]	0	[2]	[0,0]
Color gauge	$\mathbf{G}$	[1|1]	0	[0]	[1,1]

in Table I with the dimensionalities of the representations spaces

$$d_{\mathbf{SL}(\mathbb{C}^2)} = (1+2L)(1+2R), \qquad \begin{cases} d_{\mathbf{SU}(2)} = 1 + 2T \\ d_{\mathbf{SU}(3)} = (1+2C_1)(1+2C_2)(1+C_1+C_2) \end{cases}$$

The gauge interactions

$$L(\mathbf{A}_0, \vec{\mathbf{A}}, \mathbf{G}) + \mathbf{L}(\mathbf{l}, \mathbf{e}, \mathbf{q}, \mathbf{u}, \mathbf{d}) - \left(\mathbf{A}_0^k \mathbf{J}_k + \mathbf{A}_a^k \mathbf{J}_k^a + \mathbf{G}_A^k \mathbf{J}_k^A\right)$$

involve the gauge fields with their nonabelian self-interactions

$$\text{for } \mathbf{U}(1): \qquad \mathbf{L}(\mathbf{A}_0) = \mathbf{F}_{kj} \frac{\partial^k \mathbf{A}_0^j - \partial^j \mathbf{A}_0^k}{2} + g_1^2 \frac{\mathbf{F}_{kj}\mathbf{F}^{kj}}{4}$$

$$\text{for } \mathbf{SU}(2): \qquad \mathbf{L}(\vec{\mathbf{A}}) = \mathbf{F}_{kj}^c \frac{\partial^k \mathbf{A}_\kappa^j - \partial^j \mathbf{A}_\kappa^k - \epsilon_\kappa^{ab} \mathbf{A}_a^k \mathbf{A}_b^j}{2} + g_2^2 \frac{\mathbf{F}_{kj}^b \mathbf{F}_b^{kj}}{4}$$

$$\text{for } \mathbf{SU}(3): \qquad \mathbf{L}(\mathbf{G}) = \mathbf{F}_{kj}^C \frac{\partial^k \mathbf{G}_\kappa^j - \partial^j \mathbf{G}_\kappa^k - \epsilon_\kappa^{AB} \mathbf{G}_A^k \mathbf{G}_B^j}{2} + g_3^2 \frac{\mathbf{F}_{kj}^B \mathbf{F}_B^{kj}}{4}$$

and the currents of the fermion fields as 'densities' of the Lie algebra where a chiral basis

$$\gamma_k = \begin{pmatrix} 0 & \sigma_k \\ \check{\sigma}_k & 0 \end{pmatrix}$$

uses Weyl matrices $\sigma_k = (\mathbf{1}_2, \vec{\sigma})$ and $\check{\sigma}_k = (\mathbf{1}_2, -\vec{\sigma})$. $\mathbf{1}_R$ denotes the R-dimensional unit (matrix)

$$\text{for log } \mathbf{U}(1): \qquad \mathbf{J}_k = -\frac{1}{2}\mathbf{l}\check{\sigma}_k\mathbf{l}^\star - \mathbf{e}\sigma_k\mathbf{e}^\star + \frac{1}{6}\mathbf{q}\check{\sigma}_k\mathbf{q}^\star + \frac{2}{3}\mathbf{u}\sigma_k\mathbf{u}^\star - \frac{1}{3}\mathbf{d}\sigma_k\mathbf{d}^\star$$

$$\text{for log } \mathbf{SU}(2): \qquad \mathbf{J}_k^a = \mathbf{l}\check{\sigma}_k\frac{\tau^a}{2}\mathbf{l}^\star + \mathbf{q}\check{\sigma}_k\frac{\tau^a}{2}\mathbf{q}^\star$$

$$\text{for log } \mathbf{SU}(3): \qquad \mathbf{J}_k^A = \mathbf{q}\check{\sigma}_k\frac{\lambda^A}{2}\mathbf{q}^\star + \mathbf{u}\check{\sigma}_k\frac{\lambda^A}{2}\mathbf{u}^\star + \mathbf{d}\sigma_k\frac{\lambda^A}{2}\mathbf{d}^\star$$

The electromagnetic group $\mathbf{U}(1)$ is embedded into the product of the Abelian hypercharge group $\mathbf{U}(1)$ with the nonabelian isospin–color group $\mathbf{SU}(2) \times \mathbf{SU}(3)$

$$\mathbf{U}(1) \hookrightarrow \mathbf{U}(1) \circ [\mathbf{SU}(2) \times \mathbf{SU}(3)] \cong \frac{\mathbf{U}(1) \times \mathbf{SU}(2) \times \mathbf{SU}(3)}{\mathbf{I}(2) \times \mathbf{I}(3)}$$

$$\mathbf{U}(R) = \mathbf{U}(\mathbf{1}_R) \circ \mathbf{SU}(R) \cong \frac{\mathbf{U}(1) \times \mathbf{SU}(R)}{\mathbf{I}(R)}, \quad \mathbf{U}(\mathbf{1}_R) \cap \mathbf{SU}(R) \cong \mathbb{I}(R)$$

In the standard model, the representations of both factors are centrally correlated (Hucks, 1991; Saller, 1992, 1998a) via the $\mathbf{SU}(2) \times \mathbf{SU}(3)$-centrum, the cyclotomic group $\mathbb{I}(2) \times \mathbb{I}(3) = \mathbb{I}(6)$ (hexality = two-triality, 'David star'). For example, the hypercharge invariant of the left-handed quarks $\mathbf{q}$ with isospin–color multiplicity 6 is the inverse, i.e. $y = \frac{1}{6}$. The central correlation of the internal symmetries is expressed by the modulo-relations for the rational invariants $[y\|2T; 2C_1, 2C_2]$ of $\mathbf{U}(1) \circ [\mathbf{SU}(2) \times \mathbf{SU}(3)]$ carried by the standard model fermion fields

$$\begin{aligned} &6y\text{mod}2 = 2T\text{mod}2 \in \mathbb{Z}_2 \\ &6y\text{mod}3 = 2(C_1 - C_2)\text{mod}3 \in \mathbb{Z}_3, \; y \end{aligned} \quad , y \cdot d_{\mathbf{SU}(2)} \cdot d_{\mathbf{SU}(3)} \in \mathbb{Z}$$

The hypercharge as invariant of the abelian group is connected with the isospin–color representation dimension as invariant of the nonabelian group.

For the transition from interaction fields to particles (Saller, 2001b) the centrally correlated hypercharge $\mathbf{U}(1)$ and isospin $\mathbf{SU}(2)$-transformations have to be disentangled

$$\mathbf{U}(2) = \mathbf{U}(\mathbf{1}_2) \circ \mathbf{SU}(2), \quad \mathbf{U}(\mathbf{1}_2) \cap \mathbf{SU}(2) = \text{centr}\ \mathbf{SU}(2) = \{\pm\mathbf{1}_2\}$$

The transition from the two parameters $(\alpha_0, \alpha_3) \in [0, 2\pi]^2$, correlated at $(\pi, 0) \cong (0, \pi)$, to the two uncorrelated ones $(\alpha_+, \alpha_-) \in [0, 2\pi]^2$ for a maximal abelian subgroup (Cartan torus)

$$\mathbf{U}(1_2) \circ \mathbf{SO}(2) = \mathbf{U}(1)_+ \times \mathbf{U}(1)_-, \quad e^{i(1_2\alpha_0 + \tau^3\alpha_3)} = e^{i\frac{1_2+\tau^3}{2}\alpha_+} e^{i\frac{1_2-\tau^3}{2}\alpha_-}$$

is performed via the Weinberg $\mathbf{SO}(2)$-rotation ('center of charge transformation'—in analogy to the center of mass transformation in mechanics), which defines coupling constants for transformations from a direct product Cartan subgroup

$$g_1^2\frac{\mathbf{F}_0^2}{4} + g_2^2\frac{\mathbf{F}_3^2}{4} = g^2\frac{\mathbf{F}_+^2}{4} + \gamma^2\frac{\mathbf{F}_-^2}{4}$$

One direct factor $\mathbf{U}(1)_+$ is gives the electromagnetic $\mathbf{U}(1)$-action on particles. Electromagnetic relativity is described by the Goldstone manifold $\mathbf{U}(2)/\mathbf{U}(1)_+$ and experimentally visible in the three weak interactions.

The central $\mathbb{I}(2)$-correlation in the internal hypercharge–isospin group gives the necessarily integer winding numbers $z = y + T_3$ for the electromagnetic group $\mathbf{U}(1)_+$ action on the color trivial lepton particles. The remaining nonintegerness of the $\mathbf{U}(1)_+$-numbers $y + T_3 \in \{\pm\frac{1}{3}, \pm\frac{2}{3}\}$ for isospin trivial quark color triplet fields can be removed in color trivial product representations as required for hadronic particles.

The Weinberg angle θ—for the form—and the fine structure constant—for the area—determine the electroweak orthogonal triangle for the coupling constants (notation $(a^2, b^2|h^2, c^2)$ with squared lengths of the sides and the height). The orthogonal sides are the hypercharge $\mathbf{U}(1)$ and isospin $\mathbf{SU}(2)$ coupling constants and the height the electromagnetic $\mathbf{U}(1)_+$ coupling constant

$$\text{experiment: } \begin{Bmatrix} \frac{4\pi}{g^2} & \sim 137 \\ \frac{g_2^2}{g_1^2} = \cot^2\theta & \sim 3.35 \end{Bmatrix} \Rightarrow \begin{cases} \text{orthogonal triangle} \\ (g_1^2, g_2^2|g^2, \gamma^2) = g_1 g_2 \left(\frac{g_1}{g_2}, \frac{g_2}{g_1}\middle|\frac{g}{\gamma}, \frac{\gamma}{g}\right) \\ \text{with } g_1 g_2 = g\gamma,\ g_1^2 + g_2^2 = \gamma^2 \end{cases}$$

With the ground state degeneracy, implemented in the minimal standard model by a Higgs field $\langle \Phi^\star \Phi(x)\rangle = m^2$ and experimentally given by the weak interaction mass $m \sim 169\frac{\text{GeV}}{c^2}$, the electroweak triangle for the coupling constants can be written as a triangle for mass ratios, involving the masses $m_W \sim 80\frac{\text{GeV}}{c^2}$ and $m_Z \sim 91\frac{\text{GeV}}{c^2}$ for the weak charged and neutral boson

$$(g_1^2, g_2^2|g^2, \gamma^2) = \frac{2}{m^2}(m_1^2, m_W^2|m_h^2, m_Z^2) \sim \left(\frac{1}{8.4}, \frac{1}{2.5}\middle|\frac{1}{10.9}, \frac{1}{1.9}\right)$$

$$g_1 g_2 = g\gamma = \frac{2m_1 m_W}{m^2} = \frac{2m_h m_Z}{m^2} = \frac{2g^2}{\sin 2\theta} \sim \frac{1}{4.6}$$

It is this order of magnitude which should be looked for squared gauge coupling constants.

4. FREE SCATTERING STATES AND FREE PARTICLES

In quantum mechanics, e.g. for a nonrelativistic potential, there are bound states and scattering states. Free scattering states and free particles as their relativistic extension involve cyclic representations of the additive translation groups which will be considered in this section. They collect irreducible representations, e.g. for time, position, and spacetime translations $x \in \mathbb{R}^n$, $n = 1, 3, 4$. The irreducible Hilbert spaces are one dimensional, the representations are not faithful

$$\mathbb{R}^n \ni x \mapsto e^{iqx} \in \mathbf{U}(1)$$

The (energy-)momenta q as elements of the dual group characterize the irreducible representations

$$q \in \textbf{irrep}\, \mathbb{R}^n \cong \mathbb{R}^n$$

The self-dual translation representations are the direct sum of the dual irreducible ones, i.e. of a representation pair with reflected (energy-)momenta $\pm q \in \mathbb{R}^n$

$$\mathbb{R}^n \ni x \mapsto \begin{pmatrix} e^{iqx} & 0 \\ 0 & e^{-iqx} \end{pmatrix} = e^{i\sigma_3 qx} \cong e^{i\sigma_1 qx} = \begin{pmatrix} \cos qx & i \sin qx \\ i \sin qx & \cos qx \end{pmatrix} \in \mathbf{SO}(2)$$

Both $x \longmapsto e^{iqx}$ and $x \longmapsto \cos qx$ are states, the exponentials are pure states with a normalized vector $\langle q/q \rangle = 1$ from the one-dimensional Hilbert space $\mathbb{C}|q\rangle$ a pure cyclic vector. The cosine is decomposable into dual exponentials. It is the basic self-dual spherical state and characterizes an $\mathbb{R}^n$-representation on a two-dimensional Hilbert space $\mathbb{C}|q\rangle \oplus \mathbb{C}\langle q| \cong \mathbb{C}^2$ with dual basis vectors. $|q\rangle \pm \langle q| \in \mathbb{C}^2$ are cyclic vectors—not pure.

The simplest quantum mechanical example is given by the time translation representations of the harmonic oscillator with its frequency (energy) as invariant—self-dual spherical for the position–momentum pair (x, p) (cyclic vectors) and irreducible for creation and annihilation operator $(u, u^\star) = \frac{\frac{1}{\ell}x \mp i\ell p}{\sqrt{2}}$ as translation eigenvectors (pure cyclic vectors)

$$H = \frac{p^2}{2M} + k\frac{x^2}{2} = \omega\left[\ell^2 \frac{p^2}{2} + \frac{x^2}{2\ell^2}\right] = \omega\frac{\{\mathrm{u}, \mathrm{u}^\star\}}{2}$$

with frequency $\omega^2 = \frac{k}{M}$ and length $\ell^4 = \frac{1}{kM}$

$$\mathbb{R} \ni t \mapsto e^{\pm i\omega t} \in \mathbf{U}(1) \Rightarrow \begin{pmatrix} \ell x(t) \\ -i\ell p(t) \end{pmatrix} = \begin{pmatrix} \cos \omega t & i \sin \omega t \\ i \sin \omega t & \cos \omega t \end{pmatrix} \begin{pmatrix} \ell x(0) \\ -i\ell p(0) \end{pmatrix}$$

The intrinsic length ℓ will be related later to an invariant for the position representation.

In general, the positive-type functions $d \in L^\infty(\mathbb{R}^n)_+$ for cyclic translation representations are, with Bochner's (1933) theorem, Fourier transformed positive Radon measures of (energy-)momenta

$$d(x) = \int d^n q \tilde{d}(q) e^{iqx}, \tilde{d} \in \mathcal{M}(\mathbb{R}^n)_+,$$

e.g. for the irreducible and basic self-dual spherical cases mentioned earlier with a Dirac distribution

$$t \in \mathbb{R}: \quad e^{i\omega t} = \int dq\ \delta(q - \omega) e^{iqt}, \quad \cos \omega t = \int dq |\omega| \delta(q^2 - \omega^2) e^{iqt}$$

self-dual positive-type functions have Radon measures, which are symmetric under reflection $q \leftrightarrow -q (q^2$ dependent).

4.1. Euclidean Groups for Nonrelativistic Scattering

The group theoretical framework for nonrelativistic scattering is the representation theory of the semidirect Euclidean group $\mathbf{SO}(3)\vec{\times}\mathbb{R}^3$ with the rotation $\mathbf{SO}(3) \cong \mathbf{SU}(2)/\mathbb{I}(2)$ acting on the position translations $\mathbb{R}^3$. Their irreducible faithful representations are induced by irreducible $\mathbf{SO}(2) \times \mathbb{R}^3$ representations and labeled by an integer invariant n (helicity) for axial rotations $\mathbf{SO}(2)$ ('little group') around the momentum directions and the modulus of the momenta $\vec{q}^2 = P^2 > 0$ as continuous translation invariant

$$(n, P^2) \in \mathbf{irrep}[\mathbf{SU}(2)\vec{\times}\mathbb{R}^3] \cong \mathbb{N} \times \mathbb{R}_+$$

The representations are direct integrals of translation representations

$$\begin{aligned} \mathbf{SO}(3)\vec{\times}\mathbb{R}^3/\mathbf{SO}(3) \cong \mathbb{R}^3 \ni \vec{x} \mapsto d^3(\vec{x}) &= \int \frac{d^3q}{2\pi P}\delta(\vec{q}^2 - P^2)e^{-i\vec{q}\vec{x}} \\ &= \int \frac{d^2\omega}{4\pi} \cos P\vec{\omega}\vec{x} = \frac{\sin Pr}{Pr} \\ d^3(0) &= 1 \end{aligned}$$

The integration goes over the momenta on a two-sphere $\Omega^2 \cong \mathbf{SO}(3)/\mathbf{SO}(2)$ with radius $P > 0$ and directions $\vec{\omega}$

$$\vec{\omega} = \frac{\vec{q}}{|\vec{q}|} \in \Omega^2, \quad \begin{cases} \int d^2\omega = \int_0^\pi \sin\chi\, d\chi \int_{-\pi}^{\pi} d\varphi = |\Omega^2| = 4\pi \\ \delta(\vec{\omega}) = \frac{1}{\sin\chi}\delta(\chi)\delta(\varphi) \end{cases}$$

The not square integrable spherical Bessel function $\vec{x} \longmapsto \frac{\sin Pr}{Pr}$ is a state for a cyclic translation representation and a pure state for an irreducible representation of the Euclidean group.

The Hilbert space (Saller, 2005) is given by the two-sphere square integrable function (classes) $f \in L^2(\Omega^2)$ with wave packets for momentum directions

$$\langle P^2; f|P^2; f'\rangle = \int \frac{d^2\omega}{4\pi}\, \overline{f(\vec{\omega})} f'(\vec{\omega})$$

The eigenvectors are no Hilbert vectors. They constitute a distributive basis with scalar product distribution for the Hilbert space, e.g. for a representation with trivial rotation invariant (helicity) $n = 0$

distributive basis:	$\{\|P^2;\vec{\omega}\rangle \| \vec{\omega} \in \Omega^2\}$
scalar product distribution:	$\langle P^2;\vec{\omega}'\|P^2;\vec{\omega}\rangle = 4\pi\delta(\vec{\omega}-\vec{\omega}')$
translation action:	$\|P^2;\vec{\omega}\rangle \mapsto e^{iP\vec{\omega}\vec{x}}\|P^2;\vec{\omega}\rangle$
Hilbert vectors:	$\|P^2;f\rangle = \int \frac{d^2\omega}{4\pi} f(\vec{\omega})\|P^2;\vec{\omega}\rangle$
pure cyclic vectors:	$\|P^2;1\rangle = \int \frac{d^2\omega}{4\pi}\|P^2;\vec{\omega}\rangle$
	$\int \frac{d^2\omega d^2\omega'}{(4\pi)^2}\langle P^2;\vec{\omega}'\|\cos\vec{q}\vec{x}\|P^2;\vec{\omega}\rangle = \frac{\sin Pr}{Pr}$

The corresponding pure states for the representations of the Euclidean group $\mathbf{SO}(s)\vec{\times}\mathbb{R}^s$ with $s = 2, 3, \ldots$ position dimensions integrate translation representations over the momentum sphere $\Omega^{s-1} \cong \mathbf{SO}(s)/\mathbf{SO}(s-1)$. They involve Bessel functions for integer and half-integer index (more later). The translation invariant $P^2 > 0$ is used as intrinsic momentum unit

$$\begin{aligned}
\mathbb{R}^s \ni \vec{x} \mapsto d^s(\vec{x}) &= \int \frac{2d^s q}{|\Omega^{s-1}|}\delta(\vec{q}^2 - 1)e^{-i\vec{q}\vec{x}} \\
&= \int \frac{d^{s-1}\omega}{|\Omega^{s-1}|}\cos\vec{\omega}\vec{x} = \frac{2}{|\Omega^{s-1}|}\frac{\pi \mathcal{J}_{\frac{s-2}{2}}(r)}{\left(\frac{r}{2\pi}\right)^{\frac{s-2}{2}}} \\
\text{with } d^s(0) = 1, \quad \vec{\omega} &= \tfrac{\vec{q}}{|\vec{q}|} \in \Omega^{s-1}, \quad \int d^{s-1}\omega = |\Omega^{s-1}| = \frac{2\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)}
\end{aligned}$$

$L^2(\Omega^{s-1})$ is the Hilbert space with the Euclidean group action.

4.2. Poincaré Groups for Free Particles

For free relativistic particles, the nonrelativistic scattering group $\mathbf{SO}(3)\vec{\times}\mathbb{R}^3$ is embedded in the semidirect Poincaré group $\mathbf{SO}_0(1,3)\vec{\times}\mathbb{R}^4$ with the Lorentz group $\mathbf{SO}_0(1,3) \cong \mathbf{SL}(\mathbb{C}^2)/\mathbb{I}(2)$ acting on the spacetime translations $\mathbb{R}^4$. The irreducible faithful representations which are induced by irreducible $\mathbf{SU}(2) \times \mathbb{R}^4$ representations are labeled by the integer invariant spin $2J$ for rotations with fixgroup (Wigner's 'little group') $\mathbf{SU}(2)$ in a rest system and the continuous positive mass squared $m^2 > 0$ as translation invariant

$$(2J, m^2) \in \mathbf{irrep}[\mathbf{SL}(\mathbb{C}^2)\vec{\times}\mathbb{R}^4] \cong \mathbb{N} \times \mathbb{R}_+ (\text{for fixgroup } \mathbf{SU}(2))$$

With the indefinite Lorentz metric there are additional representation types for trivial mass square, e.g. for photons, and for negative mass square, not realized with particles. They are not considered here.

The Lorentz scalar matrix elements (no states) characteristic for irreducible representations, integrate spacetime translation representations

$$\begin{aligned}
\mathbf{SO}_0(1,3)\vec{\times}\mathbb{R}^4/\mathbf{SO}_0(1,3) \cong \mathbb{R}^4 \ni x \mapsto d^{(1,3)}(x) &= \int \frac{d^4q}{2\pi m^2}\delta(q^2-m^2)e^{iqx} \\
= \int \frac{d^3q}{2\pi m^2 q_0}\cos q_0x_0 \cos \vec{q}\vec{x} &= \int \frac{d^3\mathrm{y}}{2\pi}\cos m\mathrm{y}x \\
\text{with } q_0 = \sqrt{m^2+\vec{q}^2}
\end{aligned}$$

The energy–momentum ‘directions’ on the special relativity forward three-hyperboloid $\mathcal{Y}^3 \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$ for mass m^2 and positive energy can be parameterized by hyperbolic coordinates or, more familiar, by momenta

$$\mathrm{y} = \begin{pmatrix} \cosh\psi \\ \frac{\vec{q}}{|\vec{q}|}\sinh\psi \end{pmatrix} = \vartheta(q^2)\vartheta(q_0)\frac{q}{|q|} \in \mathcal{Y}^3 \quad |q| = \sqrt{|q^2|} \quad \begin{cases} \int d^3\mathrm{y} = \int_0^\infty \sinh^2\psi\, d\psi \int d^2\omega \\ \qquad = \int \frac{d^3q}{m^2q_0}|_{q_0=\sqrt{m^2+\vec{q}^2}} \\ \delta(\mathrm{y}) \quad = \frac{1}{\sinh^2\psi}\delta(\psi)\delta(\vec{\omega}) \\ \qquad = m^2q_0\delta(\vec{q}) \end{cases}$$

$$\vartheta(q^2)e^{iqx} = e^{\epsilon(q_0)i|q|\mathrm{y}x}, \quad \vartheta(q^2)\cos qx = \cos|q|\mathrm{y}x$$

The Hilbert space (Saller, 2005) is given by the free particle relevant wave packets $f \in L^2(\mathcal{Y}^3)$ for energy–momentum ‘directions’ y or for momenta $\vec{q}$, e.g. for trivial rotation invariant $J = 0$ where the elements of a distributive basis $|m_\pi^2;\vec{q}|$ can describe a pion with momentum $\vec{q}$ and positive energy $q_0 = \sqrt{m_\pi^2+\vec{q}^2}$

$$\begin{aligned}
\text{distributive basis:}\quad & \{|m^2;\mathrm{y}\rangle = |m^2;\vec{q}\rangle | \mathrm{y}\in\mathcal{Y}^3, \vec{q}\in\mathbb{R}^3\} \\
\text{scalar product distribution:}\quad & \langle m^2;\mathrm{y}'|m^2;\mathrm{y}\rangle = 2\pi\delta(\mathrm{y}-\mathrm{y}') \\
& \langle m^2;\vec{q}|m^2;\vec{q}\rangle = 2\pi m^2q_0\delta(\vec{q}-\vec{q}) \\
\text{translation action:}\quad & |m^2;\mathrm{y}\rangle \mapsto e^{im\mathrm{y}x}|m^2;\mathrm{y}\rangle \\
& |m^2;\vec{q}\rangle \mapsto e^{iqx}|m^2;\vec{q}\rangle
\end{aligned}$$

$$\text{Hilbert vectors:} \quad |m^2; f\rangle = \int \frac{d^3 y}{2\pi} f(y)|m^2; y\rangle$$
$$= \int \frac{d^3 q}{2\pi m^2 q_0} f(\vec{q})|m^2; \vec{q}\rangle$$
$$\langle m^2; f|m^2; f'\rangle = \int \frac{d^3 y}{2\pi} \overline{f(y)} f'(y)$$
$$= \int \frac{d^3 q}{2\pi m^2 q_0} \overline{f(\vec{q})} f'(\vec{q})$$

In contrast to the scattering states with compact homogenous group and finite sphere area $|\Omega^2| = 4\pi$, the integral of the distributive basis over the energy–momentum hyperboloid is not a cyclic vector. Because of the infinite volume $|\mathcal{Y}^3|$, it is a cyclic vector distribution:

$$\text{cyclic vector distribution:} \quad |m^2; 1\rangle = \int \frac{d^3 y}{2\pi} |m^2; y\rangle$$
$$\int \frac{d^3 y\, d^3 y'}{(2\pi)^2} \langle m^2; y'| \cos qx |m^2; y\rangle = \int \frac{d^4 q}{2\pi m^2} \delta(q^2 - m^2) e^{iqx}$$

The fixgroup **SO**(s)-induced representations for general Poincaré groups $\mathbf{SO}_0(1, s) \vec{\times} \mathbb{R}^{1+s}$ with $s = 1, 2, \ldots$ position dimensions integrate translation representations over the hyperboloid $\mathcal{Y}^s \cong \mathbf{SO}_0(1, s)/\mathbf{SO}(s)$. They involve Neumann functions for time-like and Macdonald functions for space-like translations, both for integer and half-integer index. The translation invariant $m^2 > 0$ is used as intrinsic mass unit

$$\mathbb{R}^{1+s} \ni x \mapsto d^{(1,s)}(x) = \int \frac{2d^{1+s} q}{|\Omega^{s-1}|} \delta(q^2 - 1) e^{iqx} = \int \frac{2d^s y}{|\Omega^{s-1}|} \cos yx$$
$$= \frac{2}{|\Omega^{s-1}|} \frac{-\vartheta(x^2)\pi \mathcal{N}_{-\frac{s-1}{2}}(|x|) + \vartheta(-x^2) 2\mathcal{K}_{\frac{s-1}{2}}(|x|)}{\left|\frac{x}{2\pi}\right|^{\frac{s-1}{2}}}$$
$$\text{with } y = \vartheta(q^2)\vartheta(q_0)\frac{q}{|q|} \in \mathcal{Y}^s, \quad |x| = \sqrt{|x^2|}$$

$L^2(\mathcal{Y}^s)$ is the Hilbert space with the Poincaré group action.

The embedded representations of the time translations and the Euclidean group

$$\mathbf{SO}_0(1, s) \vec{\times} \mathbb{R}^{1+s} \supset \mathbb{R} \times [\mathbf{SO}(s) \vec{\times} \mathbb{R}^s]$$

are seen in the partial integration decomposition, e.g. for $1 + s = 4$

$$d^{(1,3)}(x) = \int \frac{d^4q}{2\pi m^2}\delta(q^2 - m^2)e^{iqx}$$
$$= \int dq_0\vartheta\left(q_0^2 - m^2\right) 2\cos q_0x_0 \frac{\sin\sqrt{q_0^2 - m^2 r}}{m^2 r}$$

5. MEASURES AND SPACETIME COEFFICIENTS

(Energy-)momentum measures are used in the definition of free particle representations. The Lebesgue measure $\frac{d^nq}{(2\pi)^n}$ is the Plancherel measure for the irreducible translation representation $\mathbb{R}^n \ni x \mapsto e^{iqx} \in \mathbf{U}(1)$ and Haar measure d^nx. For irreducible representations of affine groups $G\vec{\times}\mathbb{R}^n$ it is modified by Dirac distributions of (energy-)momenta on homogeneous spaces G/H. They describe interaction-free structures with cyclic translation representations.

5.1. Spherical, Hyperbolic, Feynman, and Causal Measures

For the circle, one has different parameterizations, e.g.

$$\Omega^1 \ni \begin{pmatrix} q_0 \\ iq \end{pmatrix}, \qquad \text{for semicircle:} \quad \begin{pmatrix} \cos\chi \\ i\sin\chi \end{pmatrix}_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \frac{1}{\sqrt{1+p^2}}\begin{pmatrix} 1 \\ ip \end{pmatrix}_{-\infty}^{\infty}$$
$$= \frac{1}{1+v^2}\begin{pmatrix} 1-v^2 \\ 2iv \end{pmatrix}_{-1}^{1}$$

Therewith, the Euclidean group relevant measure for the momentum direction sphere, i.e. for the compact classes of orthogonal groups $\mathbf{SO}(1+s)/\mathbf{SO}(s) \cong \Omega_s$, has the parameterizations—also for $s = 0$ where applicable

$$|\Omega^s| = \int d^s\omega = \int 2d^{1+s}q\ \delta\left(q_0^2 + \vec{q}^2 - 1\right)$$
$$= \int_0^\pi (\sin\chi)^{s-1}d\chi \int d^{s-1}\omega$$
$$= \int \frac{2d^s p}{(\vec{p}^2+1)^{\frac{s+1}{2}}}$$

$$\text{polar decomposition:} \quad q = |q|\vec{\omega} \text{ with } |q|^2 = q_0^2 + \vec{q}^2, \quad \vec{\omega} \in \Omega^s$$

For noncompact classes of orthogonal groups there is the Poincaré group relevant measure of the one shell positive energy-like hyperboloid $\mathbf{SO}_0(1,s)/\mathbf{SO}(s) \cong \mathcal{Y}^s$

whose parameterizations can be obtained with the spherical–hyperbolic transition $(i\vec{q}, i\chi, i\vec{p}, i\vec{v}) \to (\vec{q}, \psi, \vec{p}, \vec{v})$

$$\begin{aligned}\int d^s y &= \int 2d^{1+s}q\ \vartheta(q_0)\delta\left(q_0^2 - \vec{q}^2 - 1\right)\\ &= \int_0^\infty (\sinh\psi)^{s-1}d\psi \int d^{s-1}\omega\\ &= \int \frac{d^2q}{\sqrt{\vec{q}^2+1}}\end{aligned}$$

$$\text{‘polar’ decomposition:}\quad q = |q|y \text{ with } |q|^2 = q_0^2 - \vec{q}^2, \quad y \in \mathcal{Y}^s$$

The Dirac ‘on-shell’ and the principal value (with $q_{\rm p}^2$) ‘off-shell’ distributions are imaginary and real part of the (anti-) Feynman distributions

$$\begin{aligned}\log(q^2 \mp io - \mu^2) &= \log|q^2 - \mu^2| \mp i\pi\vartheta(\mu^2 - q^2)\\ \tfrac{\Gamma(1+N)}{(q^2\mp io-\mu^2)^{1+N}} &= -\left(-\tfrac{\partial}{\partial q^2}\right)^{1+N}\log(q^2 \mp io - \mu^2)\\ = \left(-\tfrac{\partial}{\partial q^2}\right)^{N}\tfrac{1}{q^2\mp io-\mu^2} &= \tfrac{\Gamma(1+N)}{\left(q_{\rm p}^2-\mu^2\right)^{1+N}} \pm i\pi\delta^{(N)}(\mu^2 - q^2)\\ &\text{for } \mu^2 \in \mathbb{R} \text{ and } N = 0, 1, \ldots\end{aligned}$$

Feynman distributions are possible for any signature $\mathbf{O}(t, s)$ with positive or negative invariant μ^2.

Characteristic for and compatible only with the orthochronous Lorentz group $\mathbf{SO}_0(1, s)$ are the advanced (future) and retarded (past) causal energy–momentum distributions with positive invariant m^2 only. They are distinguished by their energy q_0 behavior

$$\begin{aligned}\log((q \mp io)^2 - m^2) &= \log|q^2 - m^2| \mp i\pi\epsilon(q_0)\vartheta(m^2 - q^2)\\ \tfrac{\Gamma(1+N)}{((q\mp io)^2-m^2)^{1+N}} &= -\left(-\tfrac{\partial}{\partial q^2}\right)^{1+N}\log((q \mp io)^2 - m^2)\\ = \left(-\tfrac{\partial}{\partial q^2}\right)^{N}\tfrac{1}{(q\mp io)^2-m^2} &= \tfrac{\Gamma(1+N)}{\left(q_{\rm p}^2-m^2\right)^{1+N}} \mp i\pi\epsilon(q_0)\delta^{(N)}(m^2 - q^2)\\ &\text{for } m^2 \geq 0 \text{ and } (q \mp io)^2 = (q_0 \mp io)^2 - \vec{q}^2\end{aligned}$$

5.2. Representation Coefficients

In this subsection, all representation matrix elements for noncompact operations are given (Vilenkin and Klimyk, 1991), which will be relevant for the spacetime theory in the following paragraphs. They can be obtained as Fourier transformed measures and involve Bessel, Neumann, and Macdonald functions.

The compact and noncompact self-dual projections of the exponential $\mathbb{C} \ni z \mapsto e^z$ come with real and imaginary complex poles $q^2 = \epsilon$

$$\int \frac{dq}{\pi} \frac{1}{q^2 - io - \epsilon} e^{iqx} = \begin{cases} i e^{i|x|}, & \epsilon = +1 \\ e^{-|x|}, & \epsilon = -1 \end{cases}$$

They define the basic self-dual spherical state $\mathbb{R} \ni x \mapsto \cos x$ and the basic self-dual hyperbolic state $\mathbb{R} \ni x \mapsto e^{-|x|}$ (more of that later).

The scalar distributions for the definite orthogonal groups in general dimension with real and imaginary singularities at $\vec{q}^2 = \pm 1$ give Bessel with Neumann and Macdonald functions respectively—wherever the Γ-functions are defined

$$m \in \mathbb{R}: \quad \int \frac{dq}{2i\pi} \frac{\Gamma(1-\nu)}{(q - io - m)^{1-\nu}} e^{iqx} = \vartheta(x) \frac{e^{imx}}{(ix)^\nu}$$

$$\begin{matrix} \mathbf{O}(s),\ s = 1, 2, 3 \ldots \\ r = \sqrt{\vec{x}^2}, \quad \nu \in \mathbb{R} \end{matrix} \begin{cases} \int \frac{d^s q}{\pi^{\frac{s}{2}-\nu}} \frac{\Gamma\left(\frac{s}{2} - \nu\right)}{(\vec{q}^2)^{\frac{s}{2}-\nu}} e^{i\vec{q}\vec{x}} = \frac{\Gamma(\nu)}{\left(\frac{r^2}{2\pi}\right)^\nu} \\ \int \frac{d^s q}{\pi^{\frac{s}{2}-\nu}} \frac{\Gamma\left(\frac{s}{2} - \nu\right)}{(\vec{q}^2 - io - 1)^{\frac{s}{2}-\nu}} e^{i\vec{q}\vec{x}} = \frac{\pi(i\mathcal{J}_\nu - \mathcal{N}_\nu)(r)}{\left(\frac{r}{2\pi}\right)^\nu} \\ \int \frac{d^s q}{\pi^{\frac{s}{2}-\nu}} \frac{\Gamma\left(\frac{s}{2} - \nu\right)}{(\vec{q}^2 + 1)^{\frac{s}{2}} - \nu} e^{i\vec{q}\vec{x}} = \frac{2\mathcal{K}_\nu(r)}{\left(\frac{r}{2\pi}\right)^\nu} \\ \qquad = e^{i\pi\nu} \frac{\pi(i\mathcal{J}_\nu - \mathcal{N}_\nu)(ir)}{\left(\frac{ir}{2\pi}\right)^\nu} \end{cases}$$

All (half-) integer index functions arise by derivation $\frac{d}{d\frac{r^2}{4\pi}} = \frac{2\pi}{r}\frac{d}{dr}$, called two-sphere spread

$$\mathbb{R}_+ \ni r \mapsto \frac{(\pi\mathcal{J}_\nu, \pi\mathcal{N}_\nu, 2\mathcal{K}_\nu)(r)}{\left(\frac{r}{2\pi}\right)^\nu} = \begin{cases} \left(-\frac{d}{d\frac{r^2}{4\pi}}\right)^N (\cos r, \sin r, e^{-r}) \\ \qquad \nu + \frac{1}{2} = N = 0, 1, 2, \ldots \\ \left(-\frac{d}{d\frac{r^2}{4\pi}}\right)^N (\pi\mathcal{J}_0(r), \pi\mathcal{N}_0(r), 2\mathcal{K}_0(r)) \\ \qquad \nu = N = 0, 1, 2, \ldots \end{cases}$$

The half-integer index start from the exponentials. The integer index Bessel functions begin with $\mathcal{J}_0$, which is used for scattering in the position plane $\mathbf{SO}(2)\vec{\times}\mathbb{R}^2$. $\mathcal{J}_0$ integrates position translation states $x \mapsto \cos px$ with the invariants on a circle $p = \cos\chi \in \Omega^1$

$$\pi\mathcal{J}_0(r) = \int d^2q\ \delta(\vec{q}^2 - 1) e^{-i\vec{q}\vec{x}} = \int_0^\pi d\chi \cos(r\cos\chi) = \pi \sum_{k=0}^\infty \frac{(-\frac{r^2}{4})^k}{(k!)^2}$$

The integer index Neumann and Macdonald functions start from free particles in rotation free two-dimensional spacetime $\mathbf{SO}_0(1,1)\vec{\times}\mathbb{R}^2$ and involve integrals of time translation states $t \mapsto \cos\omega t$ and position translation state $z \mapsto e^{-|Qz|}$ with the invariants on a hyperbola $(\omega, Q) = \cosh\psi \in \mathcal{Y}^1$

$$\int d^2q\ \delta(q^2-1)e^{iqx} = \int d\psi\ [\vartheta(x^2)\cos(|x|\cosh\psi) + \vartheta(-x^2)e^{-|x|\cosh\psi}]$$
$$= -\vartheta(x^2)\pi\mathcal{N}_0(|x|) + \vartheta(-x^2)2\mathcal{K}_0(|x|)$$
$$\begin{pmatrix} -\pi\mathcal{N}_0 \\ 2\mathcal{K}_0 \end{pmatrix}(r) = -2\sum_{k=0}^{\infty}\frac{(\mp\frac{r^2}{4})^k}{(k!)^2}\left[\log\frac{r}{2} - \Gamma'(1) - \varphi(k)\right]$$
$$\varphi(0) = 0, \quad \varphi(k) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}, \quad k = 1, 2, \ldots$$
$$-\Gamma'(1) = \lim_{k\to\infty}[\varphi(k) - \log k] = 0.5772\ldots \text{(Euler's constant)}$$

The (half-) integer index Bessel, Neumann and Macdonald functions are relevant for (odd) even dimensions and rank R

$$\mathbf{SO}_0(t,s) \text{ with } t+s = \begin{cases} \nu + \frac{3}{2} = 1 + N = 1 + 2R = 1, 3, 5, \ldots \\ \nu + 2 = 2 + N = 2R \qquad\quad = 2, 4, 6, \ldots \end{cases}$$

For $R = 1, 2, \ldots$ only the Bessel functions are regular at $r = 0$. The characteristic difference for even and odd dimension is seen in the use of the rotation free case with $\mathbb{R}$-representations for $\nu = -\frac{1}{2}$: The half-integer index functions arise by derivation, i.e. two-sphere spread, with respect to the group parameter, starting with $\nu = -\frac{1}{2}$, whereas the integer index functions start from $\nu = 0$ and involve a finite integration $\zeta \in [0, 1]$ over $\mathbb{R}$-representations

$$\text{for } \nu = -\frac{1}{2} : \ \cos\ r \begin{matrix} \nearrow \\ \searrow \end{matrix} \begin{matrix} \dfrac{\sin r}{r} = \dfrac{d}{d\frac{r^2}{2}}\cos r & \text{for } \nu = \frac{1}{2} \\ \pi\mathcal{J}_0(r) = \displaystyle\int_0^1 \frac{2d\zeta}{\sqrt{1-\zeta^2}}\cos\zeta r & \text{for } \nu = 0 \end{matrix}$$

6. BOUND STATES OF HYPERBOLIC POSITION

Self-dual spherical $\mathbf{SO}(2)$-coefficients of translation representations are states in $L^\infty(\mathbb{R})_+$ with Dirac measures in $\mathcal{M}(\mathbb{R})$

$$\mathbb{R} \ni t \mapsto \cos\omega t = \int dq|\omega|\delta(q^2 - \omega^2)e^{iqt}$$

Bound states and interactions are characterizable by self-dual hyperbolic $\mathbf{SO}_0(1,1)$-coefficients, which are square integrable states in $L^\infty(\mathbb{R})_+ \cap L^2(\mathbb{R})$ and have a rational function as positive Radon measure

$$\mathbb{R} \ni z \mapsto e^{-|Qz|} = \int \frac{dq}{\pi} \frac{|q|}{q^2 + Q^2} e^{-iqz}$$

The spherical and hyperbolic invariants come from a real and 'imaginary' momentum pair as poles in the complex momentum plane, i.e. from $q = \pm\omega$ and $q = \pm i|Q|$, respectively. In contrast to the hyperbolic state, the spherical state is decomposable into a direct sum.

The one-dimensional quantum mechanical example is given by the Schrödinger functions of the harmonic oscillator. They are position representation matrix elements with the representation invariant the inverse intrinsic length $Q^2 = \frac{1}{\ell^4} = kM$. Here the hyperbolic state $z \mapsto e^{-|Q|r}$ with positive definite coordinate shows up in a re-parametrization with the square of the usual position parameter $r = \frac{x^2}{2}$

$$\left[\ell^2 \frac{p^2}{2} + \frac{x^2}{2\ell^2}\right] \psi(x) = \frac{E}{\omega}\psi(x) \Rightarrow \psi_0(x) = e^{-\frac{x^2}{2\ell^2}} = e^{-|Q|r} \text{ with } \frac{2E_0}{\omega} = 1$$

In contrast to free states, bound states for nonabelian groups use higher order momentum poles, where the order depends on the position space dimension. This will be exemplified by the nonrelativistic hydrogen atom bound states, which represent the noncompact nonabelian group $\mathbf{SO}_0(1,3)$ and start with momentum dipoles.

6.1. The Kepler Factor and the Coulomb Potential

The bound state solutions of the nonrelativistic quantum mechanical hydrogen atom with the Coulomb potential in the Hamiltonian $H = \frac{\vec{p}^2}{2} - \frac{1}{r}$ (intrinsic units) are characterized by integers for a rank 2 invariance group: A principal quantum number $k = 1, 2, \ldots$ and quantum numbers for angular momentum $L = 0, 1, \ldots, k-1$ and its direction $L_3 \leq |L|$. The compact group invariants determine the integer degree both of the spherical harmonics Y^L and of the Laguerre polynomials L^N with the sum of the degrees $L + N = 2J = k - 1 = 0, 1, \ldots$ in the Schrödinger wave functions

$$\left[\frac{\vec{p}^2}{2} - \frac{1}{r}\right]\psi(\vec{x}) = E\psi(\vec{x}), \quad \begin{cases} |k; L, L_3\rangle \sim \psi(\vec{x}) \sim r^L\, \mathrm{Y}^L_{L_3}(\varphi,\theta)\mathrm{L}^N_{1+2L}\left(\frac{2r}{k}\right) e^{-\frac{r}{k}} \\ \qquad \sim (\vec{x})^L_{L_3}\, \mathrm{L}^N_{1+2L}\left(\frac{2r}{k}\right) e^{-\frac{r}{k}} \\ \text{with } 2E = -\frac{1}{k^2}, \quad k = L + N + 1 \end{cases}$$

It is more appropriate to combine the spherical harmonics for the two-sphere representations with the matching radial power to the harmonic polynomials $(\vec{x})^L_{L_3} \sim r^L \, \mathrm{Y}^L_{L_3}(\varphi, \theta)$ with $\vec{\partial}^2(\vec{x})^L_{L_3} = 0$, acted upon with irreducible rotation group $\mathbf{SO}(3)$-representations. The remaining factor represents the position radial variable $\mathbb{R}_+ \ni r \mapsto \mathrm{L}^N_{1+2L}(\frac{2r}{k})e^{-\frac{r}{k}}$.

The separation of the harmonic polynomials in the general Schrödinger equation with rotation symmetry leaves equations for the radial representation coefficients $r \mapsto d_L(r)$

$$\left[\frac{\vec{p}^2}{2} + V(r)\right]\psi(\vec{x}) = E\psi(\vec{x})$$

$$\psi(\vec{x}) = \sum_{L=0}^{\infty} \sum_{m=-L}^{L} (\vec{x})^L_m d_L(r) \Rightarrow \left[\frac{d^2}{dr^2} + \frac{2(1+L)}{r}\frac{d}{dr} - 2V(r) + 2E_L\right] d_L(r) = 0$$

The condition to obtain for the position representation the basic self-dual hyperbolic state $r \mapsto e^{-|Q|r}$ as $L = 0$ solution of the Schrödinger equation determines the Kepler factor as potential

$$d_0(r) = e^{-|Q|r} \;\text{ with }\; \left[\frac{d^2}{dr^2} + \frac{2}{r}\frac{d}{dr} - 2V(r) + 2E_0\right] d_0(r) = 0$$

$$\Rightarrow \;\; V(r) = -\frac{|Q|}{r} \text{ and } 2E_0 = -Q^2$$

The free hyperbolic 'wave' is the hydrogen ground state. Q^2 is the position representation characterizing invariant.

6.2. The Multipoles of the Hydrogen Atom

The hyberbolic structure of a nonrelativistic dynamics with the Coulomb–Kepler potential $\frac{1}{r}$ and the invariance of the Lenz–Runge 'perihelion' vector has been exploited quantum mechanically by Fock (1935). With the additional rotation invariance the bound state vectors come in irreducible k^2-dimensional representations of the group $\mathbf{SO}(4) = \frac{\mathbf{SU}(2)\times\mathbf{SU}(2)}{\mathbf{I}(2)}$, centrally correlating two $\mathbf{SU}(2)$s, with the integer invariant $k = 1 + 2J = 1, 2, \ldots$

The measure of the three-sphere as the manifold of the orientations of the rotation group $\mathbf{SO}(3)$ in the invariance group $\mathbf{SO}(4)$ has a momentum parametrization by a dipole

$$\Omega^3 \cong \mathbf{SO}(4)/\mathbf{SO}(3), \qquad \frac{1}{\sqrt{\vec{q}^2+1}}\begin{pmatrix}1\\ i\vec{q}\end{pmatrix} \in \Omega^3 \subset \mathbb{R}^4$$

$$\Rightarrow |\Omega^3| = \int d^3\omega = \int \frac{2d^3q}{(\vec{q}^2+1)^2} = 2\pi^2$$

Ω^3-integration of the pure translation states $\mathbb{R}^3 \ni \vec{x} \mapsto e^{-i\vec{q}\vec{x}} \in \mathbf{U}(1)$, i.e. the Fourier transformed Ω^3-measure, gives the hydrogen ground state function as a scalar representation coefficient of three-position space

$$\mathcal{Y}^3 \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3) \cong \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2) \ni \vec{x} \mapsto \int \frac{d^3q}{\pi^2} \frac{|Q|}{(\vec{q}^2 + Q^2)^2} e^{-i\vec{q}\vec{x}}$$
$$= e^{-|Q|r}$$

In the bound states, three-position space is represented as three-hyperboloid with a continuous invariant Q^2 for the imaginary 'momenta' $\vec{q}^2 = -Q^2$ on a two-sphere Ω^2 and a discrete rotation invariant $2J \in \mathbb{N}$.

The bound states are matrix elements of infinite dimensional cyclic principal $\mathbf{SL}(\mathbb{C}^2)$-representations where—with the Cartan subgroups $\mathbf{SO}(2) \times \mathbf{SO}_0(1,1)$—the irreducible ones are characterized by one integer and one continuous invariant

$$(2J, Q_\pm^2) \in \textbf{ireep SL}\,(\mathbb{C}^2) \cong \mathbb{N} \times \mathbb{R}_+ (\text{principal series})$$

In the language of induced representations, the bound states of the hydrogen atom are rotation $\mathbf{SO}(3)$-intertwiners on the group $\mathbf{SO}_0(1,3)$ ($\mathcal{Y}^3$-functions) with values in Hilbert spaces with $\mathbf{SO}(3)$-representations in $(1+2J)^2$-dimensional $\mathbf{SO}(4)$-representations.

For the nonrelativistic hydrogen atom bound states, the rotation dependence $\vec{x}$ is effected by momentum derivation of the Ω^3-measure

$$\vec{x}e^{-r} = \int \frac{d^3q}{\pi^2} \frac{4i\vec{q}}{(1+\vec{q}^2)^3} e^{-i\vec{q}\vec{x}} \text{ with } \frac{4\vec{q}}{(1+\vec{q}^2)^3} = -\frac{\partial}{\partial\vec{q}} \frac{1}{(1+\vec{q}^2)^2}$$

The three-vector factor $\frac{2\vec{q}}{1+\vec{q}^2}$ is uniquely supplemented to a normalized four-vector on the three-sphere—a parametrization of the sphere

$$\frac{1}{1+\vec{q}^2}\begin{pmatrix} \vec{q}^2 - 1 \\ 2i\vec{q} \end{pmatrix} = \begin{pmatrix} \cos\chi \\ \frac{\vec{q}}{|q|} i \sin\chi \end{pmatrix} = \begin{pmatrix} p_0 \\ i\vec{q} \end{pmatrix} \in \Omega^3 \subset \mathbb{R}^4, \quad p_0^2 + \vec{p}^2 = 1$$

The normalized four-vector

$$\mathrm{Y}^{(1,1)}(p) \sim \begin{pmatrix} p_0 \\ i\vec{p} \end{pmatrix} \in \Omega^3$$

is the analogue to the normalized three-vector

$$\mathrm{Y}^1\left(\frac{\vec{q}}{|\vec{q}|}\right) \sim \frac{\vec{q}}{|\vec{q}|} \in \Omega^2$$

used for the build-up of the two-sphere harmonics

$$\mathrm{Y}^L\left(\frac{\vec{q}}{|\vec{q}|}\right) \sim \left(\frac{\vec{q}}{|\vec{q}|}\right)^L.$$

Analogously, the higher order Ω^3-harmonics arise from the totally symmetric traceless products $\mathrm{Y}^{(2J,2J)}(p) \sim (p)^{2J}$, e.g. the nine independent components in the 4×4 matrix

$$\mathrm{Y}^{(2,2)}(p) \sim (p)^2_{jk} = p_j p_k - \frac{\delta_{jk}}{4} \cong \left(\begin{array}{c|c} \frac{3p_0^2 - p^{-2}}{4} & i p_0 p_a \\ \hline i p_0 p_b & p_a p_b - \frac{\delta_{ab}}{4} \end{array}\right)$$

$$\text{with } p_a p_b - \frac{\delta_{ab}}{4} = p_a p_b - \frac{\delta_{ab}}{3}\vec{p}^2 - \frac{\delta_{ab}}{3}\frac{3p_0^2 - \vec{p}^2}{4} \text{ for } p^2 = 1$$

The Kepler bound states in $(1+2J)^2$-multiplets come with $2J$-dependent Ω^3-harmonics and multipoles

$$\mathcal{Y}^3 \ni \vec{x} \mapsto \int \frac{d^3q}{\pi^2} \frac{1}{(1+\vec{q}^2)^2} (p)^{2j} e^{-i\vec{q}Q\vec{x}} \quad \text{with} \begin{cases} p = \frac{1}{1+q^{-2}} \begin{pmatrix} q^{-2} - 1 \\ 2i\vec{q} \end{pmatrix} \\ Q = \frac{1}{1+2J} \end{cases}$$

As illustration the $k = 2$ bound state quartet with tripole vector

$$Q = \frac{1}{2}: \quad \int \frac{d^3q}{\pi^2} \frac{1}{(1+\vec{q}^2)^2} \begin{pmatrix} p_0 \\ i\vec{p} \end{pmatrix} e^{-i\vec{q}Q\vec{x}} = \int \frac{d^3q}{\pi^2} \frac{1}{(1+\vec{q}^2)^3} \begin{pmatrix} \vec{q}^2 - 1 \\ 2i\vec{q} \end{pmatrix} e^{-i\vec{q}Q\vec{x}}$$

$$= \begin{pmatrix} \frac{1-Qr}{2} \\ \frac{Q\vec{x}}{2} \end{pmatrix} e^{-Qr} = \begin{pmatrix} \frac{1}{4}\mathrm{L}_1^1(2Qr) \\ \frac{Q\vec{x}}{2}\mathrm{L}_2^0(2Qr) \end{pmatrix}$$

and the $k = 3$ bound state nonet with quadrupole tensor measure

$$Q = \frac{1}{3}: \quad \int \frac{d^3q}{\pi^2} \frac{1}{(1+\vec{q}^2)^2} \begin{pmatrix} ep_0^2 - \vec{p}^2 \\ i p_0 \vec{p} \\ 3\vec{p} \otimes \vec{p} - \mathbf{1}_3 p^{-2} \end{pmatrix} e^{-i\vec{q}Q\vec{x}}$$

$$= \int \frac{d^3q}{\pi^2} \frac{4}{(1+\vec{q}^2)^4} \begin{pmatrix} 3\frac{(\vec{q}^2-1)^2}{2} - \vec{q}^2 \\ i\vec{q}\,\frac{\vec{q}^2-1}{2} \\ 3\vec{q} \otimes \vec{q} - \mathbf{1}_3\vec{q}^2 \end{pmatrix} e^{-i\vec{q}Q\vec{x}}$$

$$= \begin{pmatrix} 1 - 2Qr + \frac{2Q^2r^2}{3} \\ \frac{2-Qr}{3}\frac{Q\vec{x}}{2} \\ \frac{Q^2}{2}\left(\mathbf{1}_3\frac{r^2}{3} - \vec{x} \otimes \vec{x}\right) \end{pmatrix} e^{-Qr} = \begin{pmatrix} \frac{1}{3}\mathrm{L}_1^2(2Qr) \\ \frac{Q\vec{x}}{2}\frac{1}{6}\mathrm{L}_3^1(2Qr) \\ \frac{Q^2}{2}\left(\mathbf{1}_3\frac{r^2}{3} - \vec{x} \otimes \vec{x}\right)\mathrm{L}_5^0(2Qr) \end{pmatrix}$$

6.3. Yukawa Potentials as Tangent Coefficients

States for free translation representations obey homogeneous equations $(\partial^2 + P^2)d = 0$ with first-order self-dual derivatives. Bound states for a nonabelian action group where the dimension is strictly larger than the rank have higher order poles—e.g. second order for hyperbolic position space $\mathcal{Y}^3$. The related lower order poles do not characterize group representation coefficients, but representations of tangent translations for the nonabelian structure. For hyperbolic position $\mathcal{Y}^3$ the tangent translation representations are Yukawa potentials with first-order poles. For nontrivial invariant, they are related to Macdonald functions with half-integer index, which arise by two-sphere spread from the $\mathbf{SO}_0(1, 3)$-state

$$(\vec{\partial}^2 - Q^2)e^{-|Q|r} = -2|Q|\frac{e^{-|Q|r}}{r}$$

$$\mathbb{R}^3 \ni \vec{x} \mapsto 2|Q|\frac{e^{-|Q|r}}{r} = -\frac{\partial}{\partial\frac{r^2}{4}}e^{-|Q|r} = \int \frac{d^3q}{\pi^2}\frac{|Q|}{\vec{q}^2 + Q^2}e^{-i\vec{q}\vec{x}}$$

interaction coefficients for $\mathcal{Y}^3$

More on the general structure of interactions as Lie algebra representation coefficients will be given in the following section.

7. RESIDUAL REPRESENTATIONS

The method of residual representations with (energy-)momentum distributions is intended to generalize, especially to nonabelian noncompact operations, the cyclic Hilbert representations of translations via positive (energy-)momentum measures. It uses the injection of the Fourier transformed Radon measures into the essentially bounded function classes

$$\int d^nq\ e^{iqx}\mathcal{M}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n), \qquad \int d^nq\ \tilde{d}(q)e^{iqx} = d(x)$$

The form of residual representations shows a generalization and modification of the Feynman propagators as used in canonical quantum field theory.

The goal of the residual representation method is to translate the relevant representation structures of homogeneous spaces (real Lie groups) and its tangent translations (Lie algebras, more on that later)—invariants, normalizations, product representations, etc.—into the language of rational complex (energy-)momentum functions with its poles, its residues and convolution products.

Semi-simple and reductive Lie groups have factorizations $G = KP$ into maximal compact group K and parabolic subgroups (Knapp, 1986) $P = MAN$ with noncompact Cartan subgroup A, its centralizer MA and nilpotent log N. The subgroup MA is similar to the fixgroup defined direct product subgroups in affine groups $H_0 \times \mathbb{R}^n \subseteq H \vec{\times} \mathbb{R}^n$ where induced representations are used for free scattering and particle states. It remains to be seen if parabolically induced representations of G can be connected with the residual representations considered in the following sections.

7.1. Residual Representations of Symmetric Spaces

Harmonic analysis of a symmetric space G/H with real Lie groups $G \supseteq H$ analyzes complex G/H-mappings with respect to irreducible G-representations with the related invariants. The eigenvalues (weights) of the group G-representations are a subset of the linear Lie algebra forms $(\log G)^T$. For translations all linear forms are weights, the (energy-)momenta. For simple groups, the weights constitute a subset of the weight space W^T (linear forms of a Cartan Lie algebra W) with the dimension the Lie algebra rank, $\dim_{\mathbb{R}} W^T = \mathrm{rank}_{\mathbb{R}} \log G$. The weights are discrete for a compact group. The Lie algebra is acted upon with the adjoint representation of the group in the affine group $G \vec{\times} \log G$, its forms with the coadjoint (dual) one. Invariants are multilinear Lie algebra forms, e.g. linear for abelian groups or the bilinear Killing form for semi-simple groups.

The tangent spaces of G/H are isomorphic to the corresponding Lie algebra classes, denoted by $\log G/H = \log G/\log H$ with $\dim_{\mathbb{R}} \log G/H = \dim_{\mathbb{R}} G - \dim_{\mathbb{R}} H$. It inherits the adjoint action of the group G, the linear forms the coadjoint one.

Now, the definition of residual representations: Functions on (representation coefficients of) a symmetric space G/H, especially $d \in L^\infty(G)$

$$d : (G/H)_{\mathrm{repr}} \to \mathbb{C}, x \longmapsto d(x)$$

are assumed to be parameterizable by vectors $x \in V$ (translations) of an orbit in a real vector space with fixgroup H

$$x \in G \bullet x_0 \cong G/H, \quad G \bullet x_0 \subseteq V \cong \mathbb{R}^n$$

e.g., a group G by its Lie algebra $\log G$ (canonical coordinates) like $\mathbf{SU}(2) \cong \{e^{i\vec{\sigma}\vec{x}} | \vec{x} \in \mathbb{R}^3\}$ or the hyperboloid $\mathbf{SO}_0(1,3)/\mathbf{SO}(3) \cong \{x \in \mathbb{R}^4 | x^2 = \ell^2 > 0, x_0 > 0\}$ by the vectors of a time-like orbit. With the dual space $q \in V^T \cong \mathbb{R}^n$ (by abuse of language called (energy-)momenta, also in the general case), e.g. the dual Lie algebra, the representations of G/H are characterizable by G-invariants $\{I_1, \ldots, I_R\}$, with rational values for a compact and rational or continuous values for a noncompact group. The invariants are given by q-polynomials and can be built by multilinear invariants—$q = m$ for an abelian group, quadratic invariants $q^2 = \pm m^2$, e.g. Killing form invariants, etc.

If there exists a distribution of the (energy-)momenta, especially $\tilde{d} \in \mathcal{M}(\mathbb{R}^n)$, e.g. for positive-type functions, whose Fourier transformation gives the functions d on the symmetric space and if the generalized function $\tilde{d}$ comes as quotient of two polynomials where the invariant zeros of the denominator polynomial $Q(q)$ characterize a G-representation

$$\tilde{d}(q) \cong \frac{P(q)}{Q(q)} \text{ with } Q(q)\text{-factors } \{(q-m)^n, (q^2 \pm m^2)^n, (q^k \pm m^k)^n\}, m \in \mathbb{R}$$

then d is called a residual representation of G/H and the complex rational function $q \longmapsto \tilde{d}(q)$ a residual representation function (distribution).

A representation of a symmetric space G/H contains representations of subspaces K, e.g. of abelian subgroups $\mathbf{SO}(2) \subset \mathbf{SU}(2)$ or $\mathbf{SO}_0(1,1) \subset \mathbf{SL}(\mathbb{C}^2)/\mathbf{SU}(2)$. A residual G/H-representation with canonical tangent space parameters $x = (x_K, x_\perp)$ has a projection to a residual K-representation by integration $\int d^{n-s}x_\perp$ over the complementary space $\frac{\log G/H}{\log K} \cong \mathbb{R}^{n-s}$

$$K \to \mathbb{C}, \quad x_K \longmapsto d(x_K, 0) = \int \frac{d^{n-s}x_\perp}{(2\pi)^{n-s}} d(x) = \int d^s q_K \, \tilde{d}(q_K, 0) e^{iq_K x_K}$$

The integration picks up the Fourier components for trivial tangent space forms (energy–momenta) $q_\perp = 0$ of $\frac{\log G/H}{\log K}$.

A Fourier integral involves irreducible representations $x \longmapsto e^{iqx}$ of the underlying translations $x \in V$. With that, residual representations with positive distributions of the (energy-)momenta (characters) $q \in V^T$ are cyclic translation representation coefficients.

With velocities and actions measured in units $(c, \hbar)$ all energy and momentum invariants can be measured in mass units. A mass unit does not imply a translation invariant. Nontrivial invariants $m \neq 0$ can be used as intrinsic units by a rescaling of translations $x \longmapsto \frac{x}{|m|}$ and (energy-)momenta $q \longmapsto |m|q$ to obtain dimensionless Lie parameters and eigenvalues. To include the trivial case $m = 0$, invariants will be kept in most cases—and somewhat inconsequentially—in the dimensional form.

7.2. Rational Complex Representation Functions

The simplest case of residual representations is realized for time and one-dimensional position with energy and momentum distributions respectively. The representations yield—for linear invariant—matrix elements of the real one-dimensional compact and noncompact group $\mathbf{U}(1) = \exp i\mathbb{R}$ and $\mathbf{D}(1) = \exp \mathbb{R}$ respectively and—for dual invariants—of their self-dual spherical and hyperbolic doublings $\mathbf{SO}(2)$ and $\mathbf{SO}_0(1,1)$ respectively.

An irreducible $\mathbb{R}$-representation is the residue of a rational complex energy function or, equivalently, a Fourier transformed Dirac distribution supported by the linear invariant energy $m \in \mathbb{R}$

$$\mathbb{R} \ni t \longmapsto e^{imt} = \oint \frac{dq}{2i\pi} \frac{1}{q-m} e^{iqt} = \int dq\, \delta(q-m) e^{iqt} \in \mathbf{U}(1)$$

This gives the prototype of a residual representation. The integral $\oint$ circles the singularity in the mathematically positive direction.

For the abelian group $\mathbf{D}(1) \cong \mathbb{R}$, where the dimension coincides with the rank and where the eigenvalues q are the group invariants m, the transition to the residual form is a trivial transcription to the singularity $q = m$. This will be different for nonabelian groups with dimension strictly larger than rank, e.g. for the rotations $\mathbf{SO}(3)$, with dimension 3 and rank 1, with the invariant a square $\vec{q}^2 = m^2$ of the three $\mathbb{R}^3$-eigenvalues $\vec{q}$.

In the Fourier transformations of the future and past distributions, the real-imaginary decomposition into Dirac and principal value distributions goes with the order function decomposition $\vartheta(\pm t) = \frac{1 \pm \epsilon(t)}{2}$ in the functions on future $\mathbb{R}_+$ and past $\mathbb{R}_-$

$$\text{causal:} \quad \mathbb{R}_\pm \ni \vartheta(\pm t)t \longmapsto \pm \int \frac{dq}{2i\pi} \frac{1}{q \mp io - m} e^{iqt} = \vartheta(\pm t) e^{imt}$$

All those distributions originate from the same representation functions with one pole in the compactified complex plane

$$\mathbb{C} \ni q \longmapsto \frac{1}{q-m} \in \bar{\mathbb{C}}, \quad m \in \mathbb{R}$$

The position $q = m$ of the singularity is related to the continuous invariant. The Fourier transforms with different contours around the pole represent via $\vartheta(\pm t)$ the causal structure of the reals.

A representation distribution with nontrivial residue can be normalized

$$\mathbb{R} \ni 0 \longmapsto 1 = \oint \frac{dq}{2i\pi} \frac{1}{q-m} = \mathrm{res}_m \frac{1}{q-m} = \langle m|m\rangle$$

The residual normalization gives, simultaneously, both the normalization of the unit $t = 0$ representation $t \longmapsto e^{imt}$ (pure state) and the scalar product of the normalized eigenvector (pure cyclic vector) $|m\rangle$.

7.3. Compact and Noncompact Dual Invariants

Poles at dual compact representation invariants $q^2 = m^2$ can be combined from linear poles at $q = \pm|m|$, the invariants for the dual irreducible subrepresentations.

The Fourier transforms of the causal and (anti-)Feynman energy distributions are functions on the cones, the bicone and the group with $\mathbf{SO}(2)$ matrix elements

$$\text{causal:}\quad \mathbb{R}_\pm \ni \vartheta(\pm t)t \longmapsto \pm\int \frac{dq}{i\pi}\frac{\binom{q}{|m|}}{(q \mp io)^2 - m^2}e^{iqt} = \vartheta(\pm t)2\begin{pmatrix}\cos mt\\ i\sin|m|t\end{pmatrix}$$

$$\text{bicone:}\quad \mathbb{R}_+ \uplus \mathbb{R}_- \ni t \longmapsto \pm\int \frac{dq}{i\pi}\frac{\binom{|m|}{q}}{q^2 \mp io - m^2}e^{iqt} = \begin{pmatrix}1\\ \pm\epsilon(t)\end{pmatrix}e^{\pm i|mt|}$$

$$\text{group:}\quad \mathbb{R} \ni t \longmapsto \int dq\begin{pmatrix}|m|\\ q\end{pmatrix}\delta(q^2 - m^2)e^{iqt} = \begin{pmatrix}\cos mt\\ i\sin|m|t\end{pmatrix}$$

The normalization for $t = 0$ uses different matrix elements for the causal residues with two poles with equal imaginary part and for the Feynman residues with two poles with opposite imaginary part

$$\text{causal:}\quad \mathbb{C} \ni q \longmapsto \frac{q}{q^2 - m^2} \in \bar{\mathbb{C}},$$

$$\sum\oint_{\pm|m|}\frac{dq}{i\pi}\frac{q}{q^2 - m^2} = \sum \mathrm{res}_{\pm|m|}\frac{2q}{q^2 - m^2} = 2$$

$$\text{Feynman:}\quad \mathbb{C} \ni q \longmapsto \frac{|m|}{q^2 - m^2} \in \bar{\mathbb{C}},$$

$$\pm\oint_{\pm|m|}\frac{dq}{i\pi}\frac{|m|}{q^2 - m^2} = \mathrm{res}_{\pm|m|}\frac{\pm 2|m|}{q^2 - m^2} = 1$$

The functions with noncompact dual representation invariants $q^2 = -m^2$ give, as Fourier transformed Ω^1-measure, noncompact matrix elements of faithful cyclic $\mathbf{D}(1)$-representations, not irreducible

$$\mathbf{SO}_0(1,1) \cong \mathbb{R} \ni x \longmapsto \int \frac{dq}{\pi}\frac{|m|}{q^2 + m^2}e^{iqx}$$

$$= \oint \frac{dq}{2i\pi}\left[\frac{\vartheta(-x)}{q - i|m|} - \frac{\vartheta(x)}{q + i|m|}\right]e^{iqx} = e^{-|mx|}$$

The representation relevant residues are taken at imaginary ‘momenta’ $q = \pm i|m|$ in the complex momentum plane

$$\mathbb{C} \ni q \longmapsto \frac{|m|}{q^2 + m^2} \in \bar{\mathbb{C}},\quad \oint_{\pm i|m|}\frac{dq}{\pi}\frac{|m|}{q^2 + m^2} = \mathrm{res}_{\pm i|m|}\frac{\pm 2i|m|}{q^2 + m^2} = 1$$

7.4. Residual Representations of Hyperbolic Positions

Distributions of s-dimensional momenta $\vec{q} \in \mathbb{R}^s$ with the action of the rotation group $\mathbf{SO}(s)$ are used for representations (Sherman, 1975; Strichartz, 1973) of the hyperboloids $\mathcal{Y}^s$ and spheres Ω^s. For $s = 1$ 'flat' and 'hyberbolic' are isomorphic. The residual representations of nonabelian noncompact hyperboloids and compact spheres with $s \geq 2$ have to embed the nontrivial representations of the abelian groups with continuous and integer dual invariants respectively

$$\mathbf{SO}_0(1,1) \cong \mathcal{Y}^1 \ni x \longmapsto \int \frac{dq}{\pi} \frac{|m|}{q^2+m^2} e^{iqx} = e^{-|mx|}, \quad m^2 > 0$$

$$\mathbf{SO}(2) \cong \Omega^1 \ni e^{ix} \longmapsto \pm \int \frac{dq}{i\pi} \frac{|m|}{q^2 \mp io - m^2} e^{iqx} = e^{\pm i|mx|}, \quad |m| = 1, 2, \ldots$$

The pole invariants $\{\pm i|m|\}$ and $\{\pm|m|\}$ on the discrete sphere $\Omega^0 = \{\pm 1\}$ are embedded, for the nonabelian case, into singularity spheres Ω^{s-1} which arise in the Cartan factorization

$$\mathbf{SO}_0(1,s)/\mathbf{SO}(s) \cong \mathcal{Y}^s \cong \mathbf{SO}_0(1,1) \circ \Omega^{s-1}$$

$$\mathbf{SO}(1+s)/\mathbf{SO}(s) \cong \Omega^s \cong \mathbf{SO}(2) \circ \Omega^{s-1}$$

The rank of the orthogonal groups gives the real (noncompact) rank 1 for the odd-dimensional hyperboloids, i.e. one continuous noncompact invariant

$$\text{rank}_{\mathbb{R}} \mathbf{SO}_0(t,s) = R \quad \text{for} \quad t+s = 2R \quad \text{and} \quad t+s = 2R+1$$

$$\text{rank}_{\mathbb{R}} \mathbf{SO}_0(1, 2R-1) - \text{rank}_{\mathrm{R}} \mathbf{SO}(2R-1) = 1$$

$$\text{rank}_{\mathbb{R}} \mathbf{SO}_0(1, 2R) - \text{rank}_{\mathrm{R}} \mathbf{SO}(2R) = 0$$

Odd-dimensional hyperboloids and spheres, $\mathcal{Y}^s$ and Ω^s with $s = 2R - 1$, will be considered as generalization of the minimal and characteristic nonabelian case $s = 3$ with nontrivial rotations for the nonrelativistic hydrogen atom mentioned earlier.

The coefficients of residual representations of hyperboloids $\mathcal{Y}^{2R-1}$ use the Fourier transformed measure of the momentum sphere Ω^{2R-1} with singularity sphere Ω^{2R-2} for imaginary 'momenta' with continuous noncompact invariant $\vec{q}^2 = -m^2 < 0$. $\mathbf{SO}(2R)$-multiplets arise via the sphere parametrization

$$\frac{1}{\vec{q}^2+m^2} \begin{pmatrix} \vec{q}^2 - m^2 \\ 2i|m|\vec{q} \end{pmatrix} \in \Omega^{2R-1} \subset \mathbb{R}^{2R}$$

$$\text{for } \mathcal{Y}^{2R-1},\ R = 1, 2, \ldots \text{ with } \frac{2}{|\Omega^{2R-1}|} = \frac{\Gamma(R)}{\pi^R}$$

$$\vec{x} \mapsto \begin{cases} \displaystyle\int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|} \frac{|m|}{(\vec{q}^2+m^2)^R} e^{-i\vec{q}\vec{x}} = e^{-|m|r} \\ \displaystyle\int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|} \frac{R|m|}{(\vec{q}^2+m^2)^{1+R}} \begin{pmatrix} \vec{q}^2 - m^2 \\ 2i|m|\vec{q} \end{pmatrix} e^{-i\vec{q}\vec{x}} = \begin{pmatrix} R-1-|m|r \\ \vec{x} \end{pmatrix} e^{-|m|r} \end{cases}$$

Each state $\{\vec{x} \mapsto e^{-|m|r}\} \in L^\infty(\mathbf{SO}_0(1, 2R-1))_+$ with $m^2 > 0$ characterizes an infinite dimensional Hilbert space with a faithful cyclic representation of $\mathbf{SO}_0(1, 2R-1)$ as familiar for $R = 2$ from the principal series representations of the Lorentz group $\mathbf{SO}_0(1, 3)$. The positive-type function defines the Hilbert product

$$\begin{aligned}
\text{distributive basis:} & \quad \{|m^2; \vec{q}\rangle | \vec{q} \in \mathbb{R}^{2R-1}\} \\
\text{scalar product distribution:} & \quad \langle m^2; \vec{q}|m^2; \vec{q}\rangle = \tfrac{|m|}{(\vec{q}^2+m^2)^R} \tfrac{|\Omega^{2R-1}|}{2} \delta(\vec{q} - \vec{q}) \\
\text{Hilbert vectors:} & \quad |m^2; f\rangle = \int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|} f(\vec{q})|m^2; \vec{q}\rangle \\
& \quad \langle m^2; f|m^2; f'\rangle = \int \frac{2d^{2R-1}2}{|\Omega^{2R-1}|} \overline{f(\vec{q})} \frac{|m|}{(\vec{q}^2 + m^2)^R} f'(\vec{q})
\end{aligned}$$

There is a representation of each abelian noncompact subgroup in the Cartan decomposition $\mathcal{Y}^{2R-1} \cong \mathbf{SO}_0(1, 1) \circ \Omega^{2R-2}$ with the action on a distributive basis and, therewith, on the Hilbert vectors

$$\begin{aligned}
\mathbf{SO}_0(1, 1)\text{-representations for all } \vec{\omega} \in \Omega^{2R-2} : e^{-\vec{\omega}\vec{x}} \mapsto e^{-i|\vec{q}|\vec{\omega}\vec{x}} = e^{-i\vec{q}\vec{x}} \in \mathbf{U}(1) \\
\text{action of all } \mathbf{SO}_0(1, 1) : |m^2; \vec{q}\rangle \mapsto e^{-i\vec{q}\vec{x}}|m^2; \vec{q}\rangle
\end{aligned}$$

$$\begin{aligned}
\text{cyclic vector:} \quad |m^2; 1\rangle = \int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|}|m^2; \vec{q}\rangle \\
\text{with} \quad \int \frac{4d^{2R-1}q\, d^{2R-1}q'}{|\Omega^{2R-1}|^2} \langle m^2; \vec{q}|e^{-i\vec{q}\vec{x}}|m^2; \vec{q}\rangle = e^{-|m|r}
\end{aligned}$$

The scalar product is written with the positive-type function, e.g. for three-dimensional position $R = 2$ with intrinsic unit

$$\begin{aligned}
\langle f|f'\rangle = \int \frac{d^3q}{\pi^2} \overline{f(\vec{q})} \frac{1}{(\vec{q}^2 + 1)^2} f'(\vec{q}) = \int d^3x_1 d^2x_2 \overline{\tilde{f}(\vec{x}_2)} e^{-|\vec{x}_1 - \vec{x}_2|} \tilde{f}'(\vec{x}_1) \\
\text{with } f(\vec{q}) = \int d^3x\, \tilde{f}(\vec{x}) e^{i\vec{q}\vec{x}}
\end{aligned}$$

It can be brought in the form of square integrability $L^2(\mathbb{R}^3)$ by absorption of the square root

$$\psi(\vec{q}) = \frac{\sqrt{8\pi}}{\vec{q}^2 + 1} f(\vec{q}) \Rightarrow \langle f|f'\rangle = \int \frac{d^3q}{(2\pi)^3} \overline{\psi(\vec{q})} \psi'(\vec{q}) = \int d^3x \overline{\tilde{\psi}(\vec{x})} \tilde{\psi}'(\vec{x})$$

Therefore, all infinite dimensional Hilbert spaces for different continuous invariants $m^2 > 0$ are subspaces of one Hilbert space $L^2(\mathcal{Y}^{2R-1})$ with $\mathcal{Y}^{2R-1} \cong \mathbb{R}^{2R-1}$. States with different invariants are not orthogonal, i.e., they have a nontrivial Schur (1905) scalar product

$$\left\{d^{m_1^2}|d^{m_2^2}\right\} = \int d^{2R-1}x e^{-|m_1|r e^{-|m_2|r}} = \frac{|\Omega^{2R-1}|}{2\pi}\left(\frac{2}{|m_1|+|m_2|}\right)^{2R-1}$$

The orthogonality of the $\mathcal{Y}^3$-representation coefficients with different invariant $m^2 = \frac{1}{(1+2J)^2}$ in the hydrogen atom is a consequence of the different rotation invariants J.

The corresponding matrix elements of representations of odd-dimensional spheres are obtained by real–imaginary transition from noncompact to compact operations $\mathbf{SO}_0(1,1) \to \mathbf{SO}(2)$. They involve Feynman distributions with supporting singularity sphere Ω^{2R-2} for real momenta with integer compact invariant $\vec{q}^2 = m^2$, $|m| = 1, 2, \ldots$

$$\begin{matrix}\text{for } \Omega^{2R-1}: \\ R = 1, 2, \ldots\end{matrix} \vec{x} \mapsto \begin{cases} \pm \int \frac{2d^{2R-1}q}{i|\Omega^{2R-1}|} \frac{|m|}{(\vec{q}^2 \pm io - m^2)^R} e^{-i\vec{q}\vec{x}} = e^{\pm i|m|r} \\ \int \frac{d^{2R-1}}{\pi^{R-1}} |m|\delta^{(R-1)}(m^2 - \vec{q}^2) e^{-i\vec{q}\vec{x}} = \cos|m|r \end{cases}$$

The irreducible representation spaces are finite dimensional, e.g. for $R = 2$ isomorphic to $\mathbb{C}^{1+2L}$. The irreducible spaces for different discrete invariants, e.g. $L = 0, 1, \ldots$, are Schur-orthogonal subspaces (Peter and Weyl, 1927) of the infinite dimensional Hilbert space $L^2(\Omega^{2R-1})$.

The residual normalization for complex representation functions

$$\mathbb{R} \times \Omega^{2R-1} \hookrightarrow \mathbb{C} \times \Omega^{2R-1} \ni \vec{q} = |\vec{q}|\frac{\vec{q}}{|\vec{q}|} \mapsto \frac{\mu}{(\vec{q}^2 + \mu^2)^R}\bar{\mathbb{C}}, \quad \mu \in \mathbb{C}$$

has to take into account the sphere degrees of freedom in $\mathbb{C} \times \Omega^{2R-1}$, e.g. for $\mathcal{Y}^{2R-1}$

$$\text{res}_{\pm i|m|} \frac{2|m|}{(\vec{q}^2 + m^2)^R} = \int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|} \frac{|m|}{(\vec{q}^2 + m^2)^R} = \oint_{\pm i|m|} \frac{dq}{\pi} \frac{|m|(q^2)^{R-1}}{(q^2 + m^2)^R} = 1$$

The higher order q^2-power is compensated with the q^2-power of the measure. Nonscalar functions have trivial residue.

The tangent translations for the nonabelian Lie algebras $\log \mathbf{SO}(1, 2R-1)$ for the hyperboloids and $\log \mathbf{SO}(2R)$ for the spheres are represented by Yukawa potentials and spherical waves (half-integer index Macdonald and Hankel functions respectively), which arise by two-sphere spread of the states

$$R = 2, 3, \ldots, \text{ for } \mathcal{Y}^{2R-1} : \vec{x} \mapsto \int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|} \frac{1}{(\vec{q}^2 + m^2)^{R-1}} e^{-i\vec{q}\vec{x}} = 2\frac{e^{-|m|r}}{r}$$

$$\text{for } \Omega^{2R-1} : \vec{x} \mapsto \pm \int \frac{2d^{2R-1}q}{i|\Omega^{2R-1}|} \frac{1}{(\vec{q}^2 \mp io - m^2)^{R-1}} e^{-i\vec{q}\vec{x}} = 2\frac{e^{\pm|m|r}}{r}$$

8. RESIDUAL REPRESENTATIONS OF SPACETIME

The representations of the time translations $\mathbb{R}$ and of the hyperboloid $\mathcal{Y}^3$ as model of position space as seen in the nonrelativistic hydrogen bound states can be brought together in representations of homogeneous spacetime models whose matrix elements will be given in a residual formulation.

8.1. Homogeneous Causal Spacetimes

The spacetime translations in the Poincaré group $\mathbf{SO}_0(1,s)\vec{\times}\mathbb{R}^{1+s}$, $s \geq 1$, can be decomposed into Lorentz group orbits, i.e. symmetric spaces with characteristic fixgroups

$$\begin{array}{rc} \text{trivial:} & \mathbf{SO}_0(1,s)/\mathbf{SO}_0(1,s) \cong \{0\} \\ \text{time-like:} & \mathbf{SO}_0(1,s)/\mathbf{SO}(s) \cong \mathcal{Y}^s \\ \text{space-like:} & \mathbf{SO}_0(1,s)/\mathbf{SO}_0(1,s-1) \cong \mathcal{Y}^{(1,s-1)} \\ \text{light-like:} & \mathbf{SO}_0(1,1)/\{1\} \cong \mathbb{R} \\ & \mathbf{SO}_0(1,s)/\mathbf{SO}(s-1)\vec{\times}\mathbb{R}^{s-1} \cong \mathrm{V}^s,\ s \geq 2 \end{array}$$

$\mathrm{V}^s \cong \times\Omega^{s-1}$ is the tipless forward lightcone. Symmetric spaces with the same dimension $(1+s)$ as the translations and with isomorphic fixgroup for all elements are all future or all past time-like and all space-like translations. For $s \geq 2$, only the time-like ones are causally ordered. Open future $\mathbb{R}_+^{1+s}$ will be taken as the causal homogeneous model for spacetime with the dilation Poincaré group as tangent structure

$$\begin{aligned} \mathcal{D}^{1+s} &= \mathbb{R}_+^{1+s} = \{x \in \mathbb{R}^{1+s} | x^2 > 0, x_0 > 0\} \\ &\cong \mathcal{D}(1) \times \mathcal{Y}^s \\ \text{tangent } \log \mathcal{D}^{1+s} &\cong \mathbb{R}^{1+s}, \quad [\mathbf{D}(1) \times \mathbf{SO}_0(1,s)]\vec{\times}\, \mathbb{R}^{1+s} \end{aligned}$$

The fixgroup $\mathbf{SO}(s)$ is maximal compact in the reductive Lie group $\mathbf{D}(1) \times \mathbf{SO}_0(1,s)$.

The Fourier transformations of the advanced and retarded causal measures are supported by future and past respectively. They involve the off-shell principal value part

$$\mathbf{SO}_0(1,s){:} \quad \int \frac{d^{1+s}q}{\pi} \frac{1}{q_{\mathrm{P}}^2 - m^2} e^{iqx} = i\epsilon(x_0) \int d^{1+s}q\ \epsilon(q_0)\delta(m^2 - q^2) e^{iqx}$$

$$\begin{aligned} \int \frac{d^{1+s}q}{\pi} \frac{1}{(q \mp io)^2 - m^2} e^{iqx} &= \pm 2i\vartheta(\pm x_0) \int d^{1+s}q\ \epsilon(q_0)\delta(m^2 - q^2) e^{iqx} \\ &= \vartheta(\pm x_0) 2 \int \frac{d^{1+s}q}{\pi} \frac{1}{q_{\mathrm{P}}^2 - m^2} e^{iqx} \end{aligned}$$

Therewith, the characteristic function for the future cone can be written as Fourier transformed advanced causal measure with trivial invariant, characteristically different for odd and even dimensions

$R = 0, 1, \ldots,$ $\quad$ $\mathbf{SO}_0(1, 2R)$:

$$\begin{cases} \vartheta(x_0)\vartheta(x^2)2|x| = \int \frac{2d^{1+2R}q}{|\Omega^{1+2R}|} \frac{1}{[-(q-io)^2]^{1+R}} e^{iqx} \\ \vartheta(x_0)\vartheta(x^2)2\frac{x}{|x|} = \int \frac{2d^{1+2R}q}{|\Omega^{1+2R}|} \frac{iq}{[-(q-io)^2]^{1+R}} e^{iqx} \end{cases}$$

$R = 1, 2, \ldots,$ $\quad$ $\mathbf{SO}_0(1, 2R-1)$:

$$\begin{cases} \vartheta(x_0)\vartheta(x^2)2\pi = \int \frac{2d^{2R}q}{|\Omega^{2R-1}|} \frac{1}{[-(q-io)^2]^{R}} e^{iqx} \\ \vartheta(x_0)\vartheta(x^2)\pi x = \int \frac{2d^{2R}q}{|\Omega^{2R-1}|} \frac{iqR}{[-(q-io)^2]^{1+R}} e^{iqx} \end{cases}$$

The linear order function $\vartheta(\pm x_0)$ for time future and past $\mathbb{R}_\pm$ with $R = 0$ is embedded in order functions which can have Lorentz properties for $R = 1, 2, \ldots,$ e.g. scalar and vector. In the following, a shorthand notation for the characteristic future function is used

$$\vartheta(x) = \vartheta(x_0)\vartheta(x^2) \in \{0, 1\}$$

The minimal even-dimensional case $\mathcal{D}^2$ is called abelian or Cartan spacetime. Four-dimensional spacetime $\mathcal{D}^4 \supset \mathcal{D}^2$ with nontrivial rotation degrees of freedom is the minimal and characteristic nonabelian case. $\mathcal{D}^4$ can be looked at (Saller, 1997a,b, 1999) also to be the modulus set in the polar decomposition of the general linear group $\mathbf{GL}(\mathbb{C}^2) \ni g = u \circ |g|$, i.e. it parameterizes the orientation of the unitary operations $\mathbf{U}(2)$ in all complex linear operations. In addition to the future translation parameterizations, it has also an exponential parameterizations with four Lie algebra parameters

$$\mathcal{D}^4 = \{x = x_0\mathbf{1}_2 + \vec{x} = e^{\psi} | \psi = \psi_0\mathbf{1}_2 + \vec{\psi} \in \mathbb{R}^4\} \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$$

$$x = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}, \quad \psi = \begin{pmatrix} \psi_0 + \psi_3 & \psi_1 - i\psi_2 \\ \psi_1 + i\psi_2 & \psi_0 - \psi_3 \end{pmatrix}$$

$$e^{2\psi_0} = x_0^2 - r^2, \ \tanh^2|\vec{\psi}| = \frac{r^2}{x_0^2}, \frac{\vec{\psi}}{|\psi|} = \frac{\vec{x}}{r} \in \Omega^2$$

ψ_0 parameterizes eigentime e^{ψ_0}.

Even-dimensional spacetime $\mathcal{D}^{2R}$, $R = 1, 2, \ldots,$ has real rank 2, i.e. two characterizing continuous invariants for the two embedded maximal noncompact abelian operations—for causal eigentime $e^{\psi_0} \in \mathbf{D}(1)$ ('hyperbolic hopping') and for position $e^{\sigma^3\psi 3} \in \mathbf{SO}_0(1,1)$ ('hyperbolic stretching')

$$\text{Iwazawa decomposition } (G = KAN) : \mathbf{D}(1) \times \mathbf{SO}_0(1,s) = \mathbf{SO}(s) \circ \mathcal{D}^2 \circ e^{\mathbb{R}^{s-1}}$$

$$\text{with } \mathcal{D}^2 = \mathbf{D}(1) \times \mathbf{SO}_0(1,1) \ni e^{\mathbf{1}_2\psi_0+\sigma^3\psi_3} = e^{\frac{\mathbf{1}_2+\sigma^3}{2}\psi} + e^{\frac{\mathbf{1}_2-\sigma^3}{2}\psi-}$$

The two representation invariants will be introduced as masses (m_0^2, m_k^2) in a residual representation. With the representation in a unitary group, e.g.

$$\mathbf{D}(1) \times \mathbf{SO}_0(1,1) \to \mathbf{U}(\mathbf{1}_{2R}) \circ \mathbf{SO}(2R) \subset \mathbf{U}(2R)$$

there arises a real rank R-dependent correlation for both continuous invariants—in analogy to the central correlation $\mathbf{U}(1) \cap \mathbf{SU}(\mathbf{R}) \cong \mathbb{I}(R)$ of the rational hypercharge–isospin invariants in the standard model of electroweak interactions given earlier. m_0^2 will be used as intrinsic unit. The mass ratio $\kappa^2 = \frac{m_\kappa^2}{m_0^2}$ characterizes the representation and determines the gauge coupling constants (more later).

$\mathcal{D}^4$ is the nonabelian starting point also for another chain of causal symmetric spaces $\mathbf{D}(R)$ with real rank R, characterizing unitary relativity as the manifold of unitary groups $\mathbf{U}(R)$ in the general liner group $\mathbf{GL}(\mathbb{C}^R)$

$$\mathbf{D}(R) \cong \mathbf{GL}(\mathbb{C}^R)/\mathbf{U}(R), \qquad \begin{array}{c} \\ \\ \mathbf{D}(1) \subset \mathbf{D}(2) = \\ \\ \\ \\ \\ \end{array} \begin{array}{c} \mathcal{D}^2 \\ \cap \\ \mathcal{D}^4 \\ \cap \\ \mathcal{D}^6 \\ \cap \\ \cdots \end{array} \begin{array}{c} \\ \\ \subset \mathbf{D}(3) \subset \cdots \\ \\ \\ \\ \\ \end{array}$$

The causal spaces $\mathbf{D}(R)$ are real R^2-dimensional positive cones, parameterizable in the C^*-algebras of complex $R \times R$ matrices. With the determinants as $\mathbf{SL}(\mathbb{C}^R)$-invariant multilinear forms (volume forms)—for $\mathbf{D}(2)$ identical with the Lorentz metric—the future measure with invariant m_κ^R is given by

$$\text{for} \quad \mathbf{D}(R) : d^{R^2}\kappa_+(q) = \frac{d^{R^2}q}{\left[\det(q - io) - m_\kappa^R\right]^R}$$

$$= \frac{dq}{q - io - m_\kappa}, \quad \frac{d^4q}{\left[(q - io) - m_\kappa^2\right]^2}, \ldots$$

8.2. Residual Representations of Cartan Spacetime

The Lorentz compatible embedding of one-dimensional future into two-dimensional Cartan future (even-dimensional abelian spacetime) is given by the

Fourier transformed pole and dipole distribution in the scalar and vector future functions

$$\mathcal{D}^2 = \mathbb{R}^2_+ \ni \vartheta(x)x \mapsto \begin{cases} \int \frac{d^2q}{\pi} \frac{1}{-(q-io)^2} e^{iqx} = \vartheta(x)2\pi \\ \int \frac{d^2q}{\pi} \frac{iq}{[-(q-io)^2]^2} e^{iqx} = \vartheta(x)\pi x \end{cases}$$

States $z \mapsto e^{-|Qz|}$ of position $\mathcal{Y}^1 \cong \mathbf{SO}_0(1,1)$ with momentum measures $\frac{|Q|}{q_3^2+Q^2}$ are embedded with an energy-dependent invariant for the Ω^1-momentum measure

$$\frac{d^2q}{-q_{\mathrm{P}}^2+m^2} = \vartheta\left(m^2-q_0^2\right) \frac{dq_0}{|Q|} d\omega(Q) + \vartheta\left(q_0^2-m^2\right) \frac{d^2q}{-q_{\mathrm{P}}^2+m^2}$$

$$d\omega(Q) = dq_3 \frac{|Q|}{q_3^2+Q^2} \text{ with } Q^2 = m^2 - q_0^2$$

They lead to the Lorentz scalar future measures with invariant m^2. The $\mathcal{D}^2$-representation coefficients are Bessel functions.

$$\mathbb{R}^2_+ \ni \vartheta(x)x \mapsto \int \frac{d^2q}{\pi[-(q-io)^2+m^2]} e^{iqx} = \vartheta(x)2\pi \mathcal{J}_0(|mx|)$$

The projection $x = \mathbf{1}_2 t + \sigma_3 z$ on representation coefficients of position $\mathbf{SO}_0$ $(1,1)$ and of time $\mathbf{D}(1)$ is obtained by time and position integration respectively

$$\begin{aligned} \mathbf{SO}_0(1,1) \ni z \mapsto \int \frac{dt}{2\pi} \int \frac{d^2q}{\pi[-(q-io)^2+m^2]} e^{iqx} &= \int \frac{dq}{\pi} \frac{1}{q^2+m^2} e^{-iqz} \\ &= \frac{e^{-|mz|}}{|m|} \\ \mathbb{R}_+ \ni \vartheta(t)t \mapsto \int \frac{dz}{2\pi} \int \frac{d^2q}{\pi[-(q-io)^2+m^2]} e^{iqx} &= \int \frac{dq}{\pi} \frac{1}{-(q-io)^2+m^2} e^{iqt} \\ &= \vartheta(t) 2 \frac{\sin mt}{m} \end{aligned}$$

The self-dual representation of causal time $\mathbf{D}(1)$ with invariant m^2 are embedded in a Lorentz vector. It is a tangent translation distribution of spacetime $\mathcal{D}^2$

$$\begin{aligned} \mathbb{R}^2 \ni x \mapsto \int \frac{d^2q}{i\pi} \frac{q}{(q-io)^2-m^2} e^{iqx} &= \frac{\partial}{\partial x} \vartheta(x) 2\pi \mathcal{J}_0(|mx|) \\ &= \vartheta(x_0)\pi x \left[\delta_0\left(\frac{x^2}{4}\right) - m^2 \vartheta(x^2) \frac{2\mathcal{J}_1(|mx|)}{|mx|}\right] \end{aligned}$$

$$\text{with } \frac{\partial}{\partial x} = \frac{x}{2} \frac{\partial}{\partial \frac{x^2}{4}}$$

with the projections to representations of the time group and of the position tangent translations

$$\begin{aligned}
\mathbb{R}_+ \ni \vartheta(t)t \mapsto & \int \frac{dz}{2\pi} \int \frac{d^2q}{i\pi} \frac{q}{(q-io)^2 - m^2} e^{iqx} = \int \frac{dq}{i\pi} \frac{q}{(q-io)^2 - m^2} e^{iqt} \\
& = \vartheta(t) 2\cos mt \\
\mathbb{R} \ni z \mapsto & \int \frac{dt}{2\pi} \int \frac{d^2q}{i\pi} \frac{q}{(q-io)^2 - m^2} e^{iqx} = -\int \frac{dq}{i\pi} \frac{q}{q^2+m^2} e^{-iqz} \\
& = -\frac{1}{|m|} \frac{\partial}{\partial z} e^{-|mz|} = \epsilon(z) e^{-|mz|}
\end{aligned}$$

The residual Lorentz vector representation of two-dimensional spacetime are characterized by two Lorentz invariants (m_0^2, m_κ^2), which can be written with an invariant singularity integrated over a finite interval

$$\begin{aligned}
\frac{2}{q^2 - m_\kappa^2} \frac{q}{q^2 - m_0^2} &= \int_{m_\kappa^2}^{m_0^2} \frac{dm^2}{m_0^2 - m_\kappa^2} \frac{2q}{(q^2 - m^2)^2} \\
= -\frac{\partial}{\partial q} \int_{m_\kappa^2}^{m_0^2} \frac{dm^2}{m_0^2 - m_\kappa^2} \frac{1}{q^2 - m^2} &= -\frac{\partial}{\partial q} \int_0^1 d\zeta \frac{1}{q^2 - \zeta m_0^2 - (1-\zeta) m_\kappa^2} \\
&= -\frac{\partial}{\partial q} \frac{\log \frac{q^2 - m_0^2}{q^2 - m_\kappa^2}}{m_0^2 - m_\kappa^2}
\end{aligned}$$

By the Lorentz compatible embedding, both invariants contribute to the representations of both the causal group $\mathbf{D}(1)$ and the position hyperboloid $\mathbf{SO}_0(1,1)$. The Lorentz vector Cartan spacetime energy–momentum distribution

$$\text{for } \mathcal{D}^2 = \mathbf{D}(1) \times \mathbf{SO}_0(1,1) : \frac{d^2q}{i\pi \left[-(q-io)^2 + m_\kappa^2\right]} \frac{q}{(q-io)^2 - m_0^2}$$

leads to the residual representation coefficients

$$\begin{aligned}
\mathbb{R}_+^2 \ni \vartheta(x)x \mapsto & \int \frac{d^2q}{i\pi \left[-(q-io)^2 + m_\kappa^2\right]} \frac{q|m_0|}{(q-io)^2 - m_0^2} e^{iqx} \\
& = -\vartheta(x) \frac{\pi |m_0| x}{2} \frac{\partial}{\partial \frac{x^2}{4}} \frac{\mathcal{J}_0(|m_0 x|) - \mathcal{J}_0(|m_\kappa x|)}{m_0^2 - m_\kappa^2}
\end{aligned}$$

On the lightcone $x^2 = 0$, where time and position translations coincide $x^3 = \pm x^0$, the contributions from both invariants cancel each other.

The projections on representations of the causal group $\mathbf{D}(1)$ and of positive $\mathcal{Y}^1$ are

time future:

$$\mathbb{R}_+ \ni \vartheta(t)t \mapsto \int \frac{dz}{2\pi} \int \frac{d^2q}{i\pi\left[-(q-io)^2+m_\kappa^2\right]} \frac{q|m_0|}{(q-io)^2-m_0^2} e^{iqx}$$
$$= \vartheta(t)2|m_0|\frac{\cos m_0 t - \cos m_\kappa t}{m_0^2 - m_\kappa^2}$$

position:

$$\mathbf{SO}_0(1,1) \cong \mathbb{R} \ni z \mapsto \int \frac{dt}{2\pi} \int \frac{d^2q}{i\pi\left[-(q-io)^2+m_\kappa^2\right]} \frac{q|m_0|}{(q-io)^2-m_0^2} e^{iqx}$$
$$- \frac{2|m_0|}{m_0^2 - m_\kappa^2}\epsilon(z)\frac{\partial}{\partial|z|}V(|z|)$$

The position projection displays exponential interactions

$$V(|z|) = \frac{e^{-|m_\kappa z|}}{|m_\kappa|} - \frac{e^{-|m_0 z|}}{|m_0|}, \qquad \frac{\partial}{\partial|z|}V(|z|) = e^{-|m_\kappa z|} - e^{-|m_0 z|}$$

8.3. Residual Representations of Nonabelian Spacetime

Cartan spacetime is the abelian noncompact substructure of even-dimensional spacetimes

$$R = 1, 2, \ldots : \mathcal{D}^{2R} = \mathbf{D}(1) \times \mathcal{Y}^{2R-1} \cong \frac{\mathbf{D}(1) \times \mathbf{SO}_0(1, 2R-1)}{\mathbf{SO}(2R-1)}$$

Higher order poles have to be used for the states $\vec{x} \mapsto e^{-|Q|r}$ of position hyperboloids $R \geq 2$ with nontrivial rotation degrees of freedom, e.g. dipoles for four-dimensional spacetime.

$$\frac{d^2R_q}{\left(m^2-q_\mathrm{P}^2\right)^R} = \vartheta\left(m^2-q_0^2\right)\frac{d_{q_0}}{|Q|}d^{2R-1}\omega(Q) + \vartheta\left(q_0^2-m^2\right)\frac{d^{2R}q}{\left(m^2-q_\mathrm{P}^2\right)^R}$$
$$d^{2R-1}\omega(Q) = d^{2R-1}q\frac{|Q|}{(\vec{q}^2+Q^2)^R} \text{ with } Q^2 = m^2 - q_0^2$$

The Lorentz scalar causal measures of spacetime

$$\mathcal{D}^{2R} \ni \vartheta(x)x \mapsto \int \frac{2d^{2R}q}{|\Omega^{2R-1}|[-(q-io)^2+m^2]^R}e^{iqx} = \vartheta(x)2\pi\mathcal{J}_0(|mx|)$$

embed the representation of hyperbolic position

$$\mathcal{Y}^{2R-1} \ni \vec{x} \mapsto \int \frac{dt}{2\pi} \int \frac{2d^{2R}q}{|\Omega^{2R-1}|[-(q-io)^2+m^2]^R} e^{iqx}$$
$$= \int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|} \frac{1}{(\vec{q}^2+m^2)^R} e^{-i\vec{q}\vec{x}} = \frac{e^{|m|r}}{|m|}$$

with the projection on time representation coefficients (Saller, 1989)

$$\mathbb{R}_+ \ni \vartheta(t)t \mapsto \int \frac{|\Omega^{2R-1}|d^{2R-1}x}{(2\pi)^{2R}} \int \frac{2d^{2R}q}{|\Omega^{2R-1}|[-(q-io)^2+m^2]^R} e^{iqx}$$
$$= \int \frac{dq}{\pi} \frac{1}{[-(q-io)^2+m^2]^R} e^{iqt} = \vartheta(t) \frac{1}{\Gamma(R)} \left(-\frac{\partial}{\partial m^2}\right)^{R-1} \frac{2\sin mt}{m}$$
$$\text{for } R=2 = \vartheta(t) 2 \frac{\sin mt - mt\cos mt}{m^3}$$

In general, the time and position projections of even-dimensional spacetime representation coefficients, given by integer index Bessel functions, are half-integer Bessel, Neumann, and Macdonald functions.

The Lorentz vector embedded self-dual **D**(1)-representations are representations of tangent spacetime translations

$$\mathbb{R}^{2R} \ni x \mapsto \int \frac{2d^{2R}q}{i|\Omega^{2R-1}|} \frac{q}{(q-io)^2-m^2} e^{iqx} = \pi x \Gamma(R) \left(\frac{\partial}{\partial \frac{x^2}{4}}\right)^R \vartheta(x)\mathcal{J}_0(|mx|)$$

They are familiar from the interaction inducing off-shell contribution of Feynman propagators, which are proportional to Bessel functions and supported by the causal bicone

$$\int \frac{2d^{2R}q}{i|\Omega^{2R-1}|} \frac{1}{q^2-io-1} e^{iqx} = \Gamma(R)\left(\frac{\partial}{\partial \frac{x^2}{4}}\right)^{R-1} [\pi\vartheta(x^2)(i\mathcal{J}_0 - \mathcal{N}_0)(|x|)$$
$$+ \vartheta(-x^2)2\mathcal{K}_0(|x|)]$$
$$= \Gamma(R)\left(\frac{\partial}{\partial \frac{x^2}{4}}\right)^{R-1} \int d\psi [\vartheta(x^2)e^{i|x|\cosh\psi}$$
$$+ \vartheta(-x^2)e^{-|x|\cosh\psi}]$$

e.g. for four-dimensional spacetime

$$\mathbb{R}^4 \ni x \mapsto \int \frac{d^4q}{i\pi^2} \frac{q}{q_P^2 - m^2} e^{iqx} = \frac{\pi x}{2} \left[\delta'\left(\frac{x^2}{4}\right) - m^2\delta\left(\frac{x^2}{4}\right) + m^2\vartheta(x^2)\frac{4\mathcal{J}_2(|mx|)}{x^2}\right]$$

The off-shell contributions of Feynman propagators are no coefficients of Poincaré group representations.

The $\mathbf{D}(1)$-representation shows up in the time projection

$$\mathbb{R}_+ \ni \vartheta(t)t \mapsto \int \frac{|\Omega^{2R-1}|d^{2R-1}x}{(2\pi)^{2R}} \int \frac{2d^{2R}q}{i|\Omega^{2R-1}|} \frac{q}{(q-io)^2 - m^2} e^{iqx}$$
$$= \int \frac{dq}{i\pi} \frac{q}{(q-io)^2 - m^2} e^{iqt} = \vartheta(t) 2\cos mt$$

The projection on position tangent coefficients

$$\mathbb{R}^{2R-1} \ni \vec{x} \mapsto \int \frac{dt}{2\pi} \int \frac{2d^{2R}}{i|\Omega^{2R-1}|} \frac{q}{(q-io)^2 - m^2} e^{iqx} = -\int \frac{2d^{2R-1}q}{i|\Omega^{2R-1}|} \frac{\vec{q}}{\vec{q}^2 + m^2} e^{-i\vec{q}\vec{x}}$$
$$= -\frac{\partial}{\partial \vec{x}} \int \frac{2d^{2R-1}}{|\Omega^{2R-1}|} \frac{1}{\vec{q}^2 + m^2} e^{-i\vec{q}\vec{x}} = \frac{\vec{x}}{2|m|} \Gamma(R) \left(-\frac{\partial}{\partial \frac{r^2}{4}}\right)^R e^{-|m|r}$$

involves Yukawa forces, e.g. for three-dimensional position translations in four-dimensional spacetime

$$\mathbb{R}^3 \ni \vec{x} \mapsto \frac{\vec{x}}{2|m|} \left(\frac{\partial}{\partial \frac{r^2}{4}}\right)^2 e^{-|m|r} = -\vec{x} \frac{\partial}{\partial \frac{r^2}{4}} \frac{e^{-|m|r}}{r} = \frac{\vec{x}}{r} \frac{1 + |m|r}{r^2} e^{-|m|r}$$

In an $\mathbf{SO}_0(1, 2R-1)$-compatible framework with two Lorentz invariants (m_0^2, m_κ^2) both the invariants m_0^2 in the pole distribution for the Lorentz vector and the invariant m_κ^2 in the Lorentz scalar causal measure with pole of order R are used in representation of $\mathbf{D}(1)$ and $\mathcal{Y}^{2R-1}$

$$\frac{2}{\left(q^2 - m_\kappa^2\right)^R} \frac{q}{q^2 - m_0^2} = \int_{m_\kappa^2}^{m_0^2} \frac{dm^2}{m_0^2 - m^2} \left(\frac{m_0^2 - m^2}{m_0^2 - m_\kappa^2}\right)^R \frac{2qR}{(q^2 - m^2)^{1+R}}$$
$$= -\frac{\partial}{\partial q} \int_{m_\kappa^2}^{m_0^2} \frac{dm^2}{m_0^2 - m^2} \left(\frac{m_0^2 - m^2}{m_0^2 - m_\kappa^2}\right)^R \frac{1}{(q^2 - m^2)^R}$$
$$= -\frac{\partial}{\partial q} \int_0^1 d\zeta \frac{(1-\zeta)^{R-1}}{\left[q^2 - \zeta m_0^2 - (1-\zeta)m_\kappa^2\right]^R}$$

This defines the Lorentz vector advanced causal distribution as product of the Lorentz scalar measure

$$\text{for} \quad \mathcal{D}^{2R} \cong \mathbf{D}(1) \times \mathcal{Y}^{2R-1} : \ d^{2R}\kappa_+(q) \frac{q|m_0|}{(q-io)^2 - m_0^2}$$

$$\text{with} \quad d^{2R}\kappa_+(q) = \frac{2d^{2R}q}{i|\Omega^{2R-1}|\left[-(q-io)^2 + m_\kappa^2\right]^R}$$

$$\text{e.g.} \quad R = 2: \ d^4\kappa_+(q) = \frac{d^4q}{i\pi^2\left[(q-io)^2 + m_\kappa^2\right]^2}$$

with the $\mathbf{D}(1)$-related simple pole factor. The Fourier transform gives the coefficients of residual representation of even-dimensional spacetime. For example, for four-dimensional spacetime with dipole (Heisenberg, 1967) for the Lorentz scalar future measure

$$\mathbf{D}(2) = \mathcal{D}^4 = \mathbb{R}^4_+ \ni \vartheta(x)x \mapsto \int d^4\kappa_+(q)\frac{q|m_0|}{(q-io)^2 - m_0^2}e^{iqx}$$
$$= \vartheta(x)\frac{\pi|m_0|x}{2}\frac{1}{m_0^2 - m_\kappa^2}\frac{2}{\partial\frac{x^4}{4}}\left[\frac{\partial}{\partial\frac{x^2}{4}}\frac{\mathcal{J}_0(|m_0x|) - \mathcal{J}_0(|m_\kappa x|)}{m_0^2 - m_\kappa^2} - \mathcal{J}_0(|m_\kappa x|)\right]$$

the projections $x = t\mathbf{1}_2 + \vec{x}$ on representation coefficients of time future and on three-dimensional hyperbolic position

$$\text{time:}\quad \mathbb{R}_+ \ni \vartheta(t)t \mapsto \int \frac{d^3x}{8\pi^2}\int d^4\kappa_+(q)\frac{q|m_0|}{(q-io)^2 - m_0^2}e^{iqx}$$
$$= \vartheta(t)\frac{2|m_0|}{m_0^2 - m_\kappa^2}\left(\frac{\cos m_0t - \cos m_\kappa t}{m_0^2 - m_\kappa^2} + \frac{m_\kappa t\sin m_\kappa t}{2m_\kappa^2}\right)$$
$$\text{position:}\quad \mathcal{Y}^3 \ni \vec{x} \mapsto \int \frac{dt}{2\pi}\int d^4\kappa_+(q)\frac{q|m_0|}{(q-io)^2 - m_0^2}e^{iqx}$$
$$= \frac{4|m_0|\vec{x}}{\left(m_0^2 - m_\kappa^2\right)^2}\left(\frac{\partial}{\partial r^2}\right)^2 V_1(r)$$

involve Yukawa and exponential interactions

$$V_1(r) = \frac{e^{-|m_\kappa|r}}{|m_\kappa|} - \frac{e^{-|m_0|r}}{|m_0|} - \frac{m_0^2 - m_\kappa^2}{2|m_\kappa|^3}(1 + |m_\kappa|r)e^{-|m_\kappa|r}$$
$$V_3(r) = \frac{\partial}{\partial r^2}V_1(r) = \frac{e^{-|m_0|r} - e^{-|m_\kappa|r}}{r} + \frac{m_0^2 - m_\kappa^2}{2|m_\kappa|}e^{-|m_\kappa|r}$$

An exponential interaction is the two-sphere spread of a one-dimensional position representation $\frac{1}{2}e^{-r} = -\frac{\partial}{\partial\frac{r^2}{4}}(1+r)e^{-r}$ with an r-proportional contribution (Boerner, 1955; Saller, 1989) from a dipole.

9. PRODUCT REPRESENTATIONS OF SPACETIME

Product representations come with the product of representation coefficients, i.e. in a residual formulation with the convolution $*$ of (energy-)momentum distribution. The convolution picks up a residue itself

$$* \cong \delta(q_1 + q_2 - q) \cong \mathrm{res}_{q_1+q_2=q}$$

It defines the residual product, which leads to product representations. The convolution adds (energy-)momenta of singularity manifolds as imaginary and real eigenvalues for compact and noncompact representation invariants.

The Radon (energy-)momentum measures are a convolution algebra, which reflects the pointwise multiplication • property of the essentially bounded function classes

$$\int d^n q\ e^{iqx} \mathcal{M}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^n),\ \begin{cases} \mathcal{M}(\mathbb{R}^n) * \mathcal{M}(\mathcal{R}^n) & \subseteq \mathcal{M}(\mathbb{R}^n) \\ L^\infty(\mathbb{R}^n) \bullet L^\infty(\mathcal{R}^n) & \subseteq L^\infty(\mathbb{R}^n) \end{cases}$$

In the Feynman integrals of special relativistic quantum field theory as convolutions of energy–momentum distributions, the on-shell parts for translation representations give product representation coefficients of the Poincaré group, i.e. energy–momentum distributions for free states (multiparticle measures, later). The off-shell interaction contributions are not convolutable. This is the origin of the 'divergence' problem in quantum field theories with interactions. With respect to Poincaré group representations, the convolution of Feynman propagators makes no sense.

9.1. Convolutions with Linear Invariants

The convolution products for energy distributions can be read off directly from the pointwise products of representation matrix elements of time with the sum of the **D**(1)-invariants as invariant of the product

$$\begin{matrix} e^{im_1 t} \bullet e^{im_2 t} = e^{im_+ t} \\ \text{with} \quad m_+ = m_1 + m_2 \end{matrix} \Rightarrow \begin{cases} \dfrac{1}{q \mp io - m_1} \left(\pm \dfrac{*}{2i\pi}\right) \dfrac{1}{q \mp io - m_2} = \dfrac{1}{q \mp io - m_+} \\ \dfrac{1}{q - io - m_1} * \dfrac{1}{q + io - m_2} = 0 \end{cases}$$

The normalization of the residual product is the one-sphere measure as used in the residue

$$\oint \frac{dq}{2i\pi} \text{res},\ \frac{*}{2\pi} \cong \frac{1}{|\Omega^1|} \delta(q_1 + q_2 - q)$$

The residual product for the two causal function algebras, conjugated and orthogonal to each other, and the Dirac convolution algebra is summarized with the residually normalized representation functions and the integration contours

$$\vartheta(\pm t)e^{imt} = \pm \int \frac{dq}{2i\pi} \frac{1}{q \mp io - m} e^{iqt}$$

causal time $\mathbf{D}(1)$ and energies $\mathbb{R}$
$(\overset{1}{*}, q) = (\pm\frac{*}{2i\pi}, q \mp io)$ causal, orthogonal
$\frac{1}{q-m_1} \overset{1}{*} \frac{1}{q-m_2} = \frac{1}{q-m_+}$
$\delta(q-m_1) * \delta(q-m_2) = \delta(q-m_+)$

9.2. Convolutions with Self-Dual Invariants

The causal distributions with compact dual invariants

$$\pm\frac{1}{i\pi}\frac{q}{(q \pm io)^2 - m^2} = |m|\delta(q^2 - m^2) \pm \frac{1}{i\pi}\frac{q}{q_{\mathrm{p}}^2 - m^2}$$
$$= \pm\frac{1}{2i\pi}\left(\frac{1}{q \mp io - |m|} + \frac{1}{q \mp io + |m|}\right)$$

keep the property to constitute orthogonal convolution algebras, conjugated to each other

$$2\cos m_1 t \bullet 2\cos m_2 t = 2\cos m_+ t + 2\cos m_- t \text{with } m_\pm = |m_1| \pm |m_2|$$

$$\vartheta(\pm t)2\cos mt = \pm \int \frac{dq}{i\pi}\frac{q}{(q \mp io)^2 - m} e^{iqt}$$

causal time $\mathbf{D}(1)$ and energies $\mathbb{R}$
$(\overset{1}{*}, q^2) = (\pm\frac{*}{i\pi}, (q \mp io)^2)$ causal, orthogonal
$\frac{q}{q^2-m_1^2} \overset{1}{*} \frac{q}{q^2-m_2^2} = \frac{q}{q^2-m_+^2} + \frac{q}{q^2-m_-^2}$

The residual normalization for self-dual invariants uses half the one-sphere measure

$$\frac{*}{\pi} \cong \frac{2}{|\Omega^1|}\delta(q_1 + q_2 - q)$$

Since the Feynman energy distributions combine future and past distributions

$$\pm\frac{1}{i\pi}\frac{|m|}{q^2 \mp io - m^2} = |m|\delta(q^2 - m^2) \pm \frac{1}{i\pi}\frac{|m|}{q_{\mathrm{P}}^2 - m^2}$$
$$= \pm\frac{1}{2i\pi}\left(\frac{1}{q \mp io - |m|} - \frac{1}{q \pm io + |m|}\right)$$

they constitute convolution algebras, conjugated to each other, however not orthogonal $e^{+i|m_1 t|} \bullet e^{-i|m_2 t|} \neq 0$

$$e^{\pm i|mt|} = \pm \int \frac{dq}{i\pi} \frac{|m|}{q^2 \mp io - m^2} e^{iqt}$$

bicone time $\mathbb{R}_+ \uplus \mathbb{R}_-$ and energies $\mathbb{R}$
$(\overset{1}{*}, q^2) = (\pm\frac{*}{i\pi}, (q^2 \mp io)$ Feynman, not orthogonal
$\frac{\|m_1\|}{q^2 - m_1^2} \overset{1}{*} \frac{\|m_2\|}{q^2 - m_2^2} = \frac{\|m_+\|}{q^2 - m_+^2}$

The faithful Hilbert representations of $\mathcal{Y}^1$ (one-dimensional abelian position) with Fourier transformed Ω^1-measures and noncompact dual invariants constitute a convolution algebra

$$e^{-|mt|} = \int \frac{dq}{\pi} \frac{|m|}{q^2 + m^2} e^{iqz}$$

position $\mathcal{Y}^1$ and 'momenta' $\mathbb{R}$
$\|\Omega^1\| = 2\pi, \overset{1}{*} = \frac{*}{\pi}$
$\frac{\|m_1\|}{q^2 + m_1^2} \overset{1}{*} \frac{\|m_2\|}{q^2 + m_2^2} = \frac{\|m_+\|}{q^2 + m_+^2}$

9.3. Product Representations of Free Particles

Interaction free product structures have to convolute Dirac distributions for cyclic translation representations.

In contrast to the convolution of Dirac distributions for self-dual invariants $m_{1,2}^2 > 0$ with basic self-dual spherical two-dimensional representations

$$\begin{aligned}\text{abelian } \mathbb{R}: \quad & 2|m_1|\delta\left(q^2 - m_1^2\right) * 2|m_2|\delta\left(q^2 - m_2^2\right) \\ &= 2|m_-|\delta\left(q^2 - m_-^2\right) + 2|m_+|\delta\left(q^2 - m_+^2\right)\end{aligned}$$

the convolution of Dirac distributions for the infinite-dimensional representations of the Euclidean groups, $s \geq 2$, with the sphere radii as momentum invariants $q^{-2} = m^2 > 0$ is nontrivial for all momentum invariants between the 'sum and difference sphere' of the factors $m_+^2 \geq \vec{q}^2 \geq m_-^2$

$$\mathbf{SO}(s)\vec{\times}\mathbb{R}^s: \delta\left(\vec{q}^2 - m_1^2\right) \frac{*}{|\Omega^{s-2}|} \delta\left(\vec{q}^2 - m_2^2\right) = \frac{2|Q|^{s-3}}{|\vec{q}|} \vartheta\left(m_+^2 - \vec{q}^2\right) \vartheta\left(\vec{q}^2 - m_-^2\right)$$

$$s = 2, 3, \ldots$$

There arises a momentum-dependent normalization factor $|Q|$, which contains the characteristic two-particle convolution function

$$Q^2 = \frac{-\Delta\left(\vec{q}^2\right)}{4\vec{q}^2}, \quad \Delta\left(\vec{q}^2\right) = \Delta\left(\vec{q}^2, m_1^2, m_2^2\right) = \left(\vec{q}^2 - m_+^2\right)\left(\vec{q}^2 - m_-^2\right)$$

It is symmetric in the three invariants involved

$$\Delta(a, b, c) = a^2 + b^2 + c^2 - 2(ab + ac + bc) = (a + b - c)^2 - 4ab$$

The minimal cases $s = 2, 3$ are characteristic for even- and odd-dimensional position

$$\mathbf{SO}(3)\vec{\times}\mathbb{R}^3: \quad \delta\left(\vec{q}^2 - m_1^2\right)\frac{*}{2\pi}\delta\left(\vec{q}^2 - m_2^2\right) = \frac{2}{|\vec{q}|}\vartheta\left(m_+^2 - \vec{q}^2\right)\vartheta\left(\vec{q}^2 - m_-^2\right)$$

Product representations of Poincaré groups with the hyberboloid ‘radii’ as energy–momentum invariants $q^2 = m^2 \geq 0$

$$\begin{aligned}\mathbf{SO}_0(1, s)\vec{\times}\mathbb{R}^{1+s} : \vartheta(\pm q_0)\delta\left(q^2 - m_1^2\right) &\frac{*}{|\Omega^{s-1}|}\vartheta(\pm q_0)\delta\left(q^2 - m_2^2\right)\\ s = 1, 2, 3, \ldots &= \frac{2|Q|^{s-2}}{|q|}\vartheta(\pm q_0)\vartheta\left(q^2 - m_+^2\right)\end{aligned}$$

involve the two-particle threshold factor

$$Q^2 = \frac{\Delta\left(q^2\right)}{4|q|^2}, \quad \Delta\left(q^2\right) = \Delta\left(q^2, m_1^2, m_2^2\right) = \left(q^2 - m_+^2\right)\left(q^2 - m_-^2\right)$$

The minimal cases $s = 1, 2$ are characteristic for even- and odd-dimensional spacetime. $1 + s = 4$ is the minimal case with nonabelian rotations

$$\begin{aligned}\mathbf{SO}_0(1, 3)\vec{\times}\mathbb{R}^4 : \vartheta(\pm q_0)\delta\left(q^2 - m_1^2\right) &\tfrac{*}{4\pi}\vartheta(\pm q_0)\delta\left(q^2 - m_2^2\right)\\ &= \tfrac{2|Q|}{|q|}\vartheta(\pm q_0)\vartheta\left(q^2 - m_+^2\right)\end{aligned}$$

For nontrivial position, the convolution of s-dimensional on-shell hyperboloids (particle measures) does not lead to s-dimensional on-shell hyperboloids $\delta(q^2 - m_+^2)$. The squared sum of the invariants as product invariant gives the threshold for energy–momenta $q^2 = (q_1 + q_2)^2 \geq m_+^2$ in the two-particle product measure. Here, the energy is enough to produce two free particles with masses $m_{1,2}$ and momentum $(\vec{q}_1 + \vec{q}_2)^2 \geq 0$.

9.4. Product Representations of Hyperbolic Positions

The residual representations of three-dimensional hyperbolic position $\mathcal{Y}^3$ use the Fourier transformed Ω^3-measure, familiar from the nonrelativistic hydrogen

Schrödinger functions. The radii of the 'momentum' spheres as invariants are added up in the convolution

$$e^{-|m|r} = \int \frac{d^3q}{\pi^2} \frac{|m|}{(\vec{q}^2 + m^2)^2} e^{-\vec{q}\vec{x}}$$

position $\mathcal{Y}^3 \cong \mathbf{SO}_0(1,3)/\mathbf{SO}(3)$ and 'momenta' $\mathbb{R}^3$ with $\mathbf{SO}(3)$
$\lvert\Omega^3\rvert = 2\pi^2,\ \overset{3}{*} = \frac{*}{\pi^2}$
$\frac{\lvert m_1\rvert}{(\vec{q}^2 + m_1^2)^2} \overset{3}{*} \frac{\lvert m_2\rvert}{(\vec{q}^2 + m_2^2)^2} = \frac{\lvert m_+\rvert}{(\vec{q}^2 + m_+^2)^2}$

In general, the representations of odd-dimensional hyperboloids $\mathcal{Y}^{2R-1}$ come with Fourier transformed Ω^{2R-1}-measures and imaginary singularity sphere Ω^{2R-2} for the 'momentum' eigenvalues. The sphere measures can be obtained by invariant momentum derivatives

$$\frac{1}{\Gamma(R)} \left(-\frac{\partial}{\partial \vec{q}^2}\right)^{R-1} \frac{|m|}{\vec{q}^2 + m^2} = \frac{|m|}{(\vec{q}^2 + m^2)^R}, \quad R = 1, 2, \ldots$$

Product representations arise by the convolution with the sphere volume as residual normalization

$$e^{-|m|r} = \int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|} \frac{|m|}{(\vec{q}^2 + m^2)^R} e^{-i\vec{q}\vec{x}}$$

position $\mathcal{Y}^{2R-1} \cong \mathbf{SO}_0(1, 2R-1)/\mathbf{SO}(2R-1),\ R = 1, 2, \ldots$ and 'momenta' $\mathbb{R}^{2R-1}$ with $\mathbf{SO}\ (2R-1)$
$\lvert\Omega^{2R-1}\rvert = \frac{2\pi^R}{\Gamma(R)},\ \overset{2R-1}{*} = \frac{*2}{\lvert\Omega^{2R-1}\rvert}$
$\left(\frac{\partial}{\partial \vec{q}}\right)^{L_1} \frac{\lvert m_1\rvert}{(\vec{q}^2 + m_1^2)^R} \overset{2R-1}{*} \left(\frac{\partial}{\partial \vec{q}}\right)^{L_2} \frac{\lvert m_2\rvert}{(\vec{q}^2 + m_2^2)^R} = \left(\frac{\partial}{\partial \vec{q}}\right)^{L_1+L_2} \frac{\lvert m_+\rvert}{(\vec{q}^2 + m_+^2)^R}$ for $L = 0, 1, \ldots$

The convolution may involve tensor products for $\mathbf{SO}(2R-1)$-representations. In general, nontrivial $\mathbf{SO}_0(t,s)$-properties are effected by the convolution compatible (energy-)momentum derivatives

$$\frac{\partial}{\partial q} = 2q\frac{\partial}{\partial q^2}, \frac{\partial}{\partial q} \otimes q = \mathbf{1}_{t+s} + q \otimes q2\frac{\partial}{\partial q^2}$$

$$\frac{\partial}{\partial q} \otimes \frac{q}{\partial q} = \left(\mathbf{1}_{t+s} + q \otimes q2\frac{\partial}{\partial q^2}\right)2\frac{\partial}{\partial q^2}, \ldots$$

which—acting on multipoles—raise the pole order

$$-\frac{\partial}{\partial q}\frac{\Gamma(R)}{(q^2+\mu^2)^R} = \frac{2q\Gamma(1+R)}{(q^2+\mu^2)^{1+R}}$$

The representations of odd-dimensional spheres use a singularity sphere Ω^{2R-2} with real momentum eigenvalues in the convolutions

$$e^{\pm i|m|r} = \pm\int \frac{2d^{2R-1}q}{i|\Omega^{2R-1}|}\frac{|m|}{(\vec{q}^2 \mp io - m^2)^R}e^{-i\vec{q}\vec{x}}$$

sphere $\Omega^{2R-1} \cong \mathbf{SO}(2R)/\mathbf{SO}(2R-1)$, $R = 1, 2, \ldots$ and momenta $\mathbb{R}^{2R-1}$ with $\mathbf{SO}\ (2R-1)$
$|\Omega^{2R-1}| = \frac{2\pi^R}{\Gamma(R)}, (\overset{2R-1}{*}, \vec{q}^2) = \left(\pm\frac{*2}{i|\Omega^{2R-1}|}, \vec{q}^2 \mp io\right)$ not orthogonal
$\left(\frac{\partial}{\partial\vec{q}}\right)^{L_1}\frac{|m_1|}{(\vec{q}^2+m_1^2)^R} \overset{2R-1}{*} \left(\frac{\partial}{\partial\vec{q}}\right)^{L_2}\frac{|m_2|}{(\vec{q}^2+m_2^2)^R} = \left(\frac{\partial}{\partial\vec{q}}\right)^{L_1+L_2}\frac{|m_+|}{(\vec{q}^2+m_+^2)^R}$ for $L = 0, 1, \ldots$

The abelian case $R = 1$ with the one-sphere $\Omega^1 \cong \mathbf{SO}(2)$ has been used earlier for compact time $\mathbb{R}$-representations by Feynman distributions. For $R = 2$ there arise representations of the spin group $\Omega^3 \cong \mathbf{SU}(2)$.

9.5. Residual Products for Spacetime

The convolution product of causal and Feynman distributions on even-dimensional spacetimes can be computed with the familiar methods. For the causal distributions, the product residue is defined as causal too.

The convolutions of Cartan energy–momentum pole distributions are

spacetime $\mathcal{D}^2 = \mathbf{D}(1) \times \mathcal{Y}^1$ with $\mathbf{SO}_0(1,1)$, $|\Omega^1| = 2\pi$
$\left(\overset{2}{*}, q^2\right) = \begin{cases} \left(\pm\frac{*}{2i\pi}, (q \mp io)^2\right), & \text{causal, orthogonal} \\ \left(\pm\frac{*}{i\pi}, q^2 \mp io\right), & \text{Feynman, not orthogonal}\end{cases}$
$\frac{1}{q^2-m_1^2} \overset{2}{*} \frac{1}{q^2-m_2^2} = \int_0^1 d\zeta \frac{1}{\zeta(1-\zeta)q^2 - \zeta m_1^2 - (1-\zeta)m_2^2}$

The residual products display pole distributions only before the integration over finite $\zeta \in [0, 1]$, characteristic for even-dimensional spaces

$$\int_0^1 d\zeta \frac{1}{\zeta(1-\zeta)(q^2 \mp io) - \zeta m_1^2 (1-\zeta) m_2^2}$$
$$= \frac{2}{\sqrt{|\Delta(q^2)|}} \left[\mp i\pi\vartheta\left(q^2 - m_+^2\right) - \vartheta(\Delta(q^2)) \log \left| \frac{\sum(q^2) - \sqrt{4\Delta(q^2)}}{m_+^2 - m_-^2} \right| \right.$$
$$\left. - \vartheta(-\Delta(q^2)) \arctan \frac{\sqrt{-4\Delta(q^2)}}{\sum(q^2)} \right]$$

with $\quad \Delta(q^2) = \left(q^2 - m_+^2\right)\left(q^2 - m_-^2\right), \qquad \sum(q^2) = \left(q^2 - m_+^2\right) + \left(q^2 - m_-^2\right)$

The pole distributions can be written with spectral functions, e.g.

$$\int_0^1 \frac{d\zeta}{q^2 - \zeta m_0^2 - (1-\zeta) m_\kappa^2} = \int_{m_\kappa^2}^{m_0^2} \frac{dM^2}{m_0^2 - m_\kappa^2} \frac{1}{q^2 - M^2}$$
$$\int_0^1 \frac{d\zeta}{\zeta q^2 - m^2} = \int_{m^2}^{\infty} \frac{dM^2}{M^2} \frac{1}{q^2 - M^2}$$
$$\int_0^1 \frac{d\zeta(1-\zeta)}{\zeta q^2 - m^2} = \int_{m^2}^{\infty} \frac{dM^2}{M^2} \frac{M^2 - m^2}{M^2} \frac{1}{q^2 - M^2}$$

The ζ-dependent q^2-singularities disappear after ζ-integration, there arise logarithms, no energy–momentum poles. The logarithm is typical for a finite integration (Behnke, 1962), e.g. for a function holomorphic on the integration curve

$$\int_a^b dz\, f(z) = \sum \text{res} \left[f(z) \log \frac{z-b}{z-a} \right],$$
$$\begin{cases} \int_a^\infty dz\, f(z) = -\sum \text{res} \left[f(z) \log(z-a) \right] \\ \int dz\, f(z) = \sum \text{res}\, f(z) \end{cases}$$

with the sum of all residues in the closed complex plane, cut along the integration curve, here

$$\int_0^1 \frac{d\zeta}{q^2 - \zeta m_0^2 - (1-\zeta) m_\kappa^2} = \sum \text{res} \left[\frac{1}{q^2 - \zeta m_0^2 - (1-\zeta) m_\kappa^2} \log \frac{\zeta - 1}{\zeta} \right]$$
$$= \frac{1}{m_0^2 - m_\kappa^2} \log \frac{m_0^2 - q^2}{m_\kappa^2 - q^2}$$

$$\int_0^1 \frac{d\zeta}{\zeta q^2 - m^2} = \sum \text{res} \left[\frac{1}{\zeta q^2 - m^2} \log \frac{\zeta - 1}{\zeta} \right]$$
$$= \frac{\log\left(1 - \frac{q^2}{m^2}\right)}{q^2}$$
$$\int_0^1 \frac{d\zeta(1-\zeta)}{\zeta q^2 - m^2} = \sum \text{res} \frac{1-\zeta}{\zeta q^2 - m^2} \log \frac{\zeta - 1}{\zeta}$$
$$= \frac{\left(1 - \frac{m^2}{q^2}\right) \log(1 - \frac{q^2}{m^2}) - 1}{q^2}$$

In the third case, there is a nontrivial residue at the holomorphic point $\zeta = \infty$.

The corresponding structures for Minkowski spacetime as minimal case with nontrivial rotation degrees of freedom are

spacetime $\mathcal{D}^4 = \mathbf{D}(2) \times \mathcal{Y}^3$ with $\mathbf{SO}_0(1,3)$, $\lvert\Omega^3\rvert = 2\pi^2$
$(\overset{4}{*}, q^2) = \begin{cases} (\mp\frac{*}{2i\pi^2}, (q \mp io)^2), & \text{causal, orthogonal} \\ (\mp\frac{*}{i\pi^2}, q^2 \mp io), & \text{Feynman, not orthogonal} \end{cases}$
$\frac{1}{(q^2 - m_1^2)^2} \overset{4}{*} \frac{1}{(q^2 - m_2^2)^2} = \int_0^1 d\zeta \frac{\zeta(1-\zeta)}{[\zeta(1-\zeta)q^2 - \zeta m_1^2 - (1-\zeta)m_2^2]^2}$

In general, one obtains the even-dimensional spacetime $\mathcal{D}^{2R}$ distributions of energy–momenta by derivations

$$\frac{1}{\Gamma(R)} \left(-\frac{\partial}{\partial q^2} \right)^{R-1} \frac{1}{q^2 - m^2} = \frac{1}{(q^2 - m^2)^R}, \quad R = 1, 2, \ldots$$

For the Feynman distributions, the residual normalization is half the volume $\frac{\lvert\Omega^{2R-1}\rvert}{2}$ of the position-related sphere with the sign $(-1)^R$. For the causal distributions, the full volume is taken

$$\int \frac{2(-1)^R d^{2R} q}{\lvert\Omega^{2R-1}\rvert \, [(q - io)^2 - m^2]^R} e^{iqx} = \vartheta(x) 2\pi \mathcal{J}_0(\lvert mx \rvert)$$

spacetime $\mathcal{D}^{2R} = \mathbf{D}(1) \times \mathcal{Y}^{2R-1}$, $R = 1, 2, \ldots$ with $\mathbf{SO}_0(1, 2R-1)$, $\lvert\Omega^{2R-1}\rvert = \frac{2\pi^R}{\Gamma(R)}$
$\left(\overset{2R}{*}, q^2\right) = \begin{cases} \left(\mp\frac{*(-1)^R}{i\lvert\Omega^{2R-1}\rvert}, (q \mp io)^2\right), & \text{causal, orthogonal} \\ \left(\mp\frac{*2(-1)^R}{i\lvert\Omega^{2R-1}\rvert}, q^2 \mp io\right), & \text{Feynman, not orthogonal} \end{cases}$
$\left(\frac{\partial}{\partial_q}\right)^{L_1} \frac{1}{(q^2-m_1^2)^R} \overset{2R}{*} \left(\frac{\partial}{\partial_q}\right)^{L_2} \frac{1}{(q^2-m_2^2)^R} = \left(\frac{\partial}{\partial q}\right)^{L_1+L_2} \frac{1}{\Gamma(R)} \left(-\frac{\partial}{\partial q^2}\right)^{R-1} \int_0^1 d\zeta \frac{1}{\zeta(1-\zeta)q^2 - \zeta m_1^2 - (1-\zeta)m_2^2}$

with the examples—wherever the Γ-functions are defined for $\nu \in \mathbb{R}$

$$\frac{\Gamma(R+\nu_1)}{(q^2-m_1^2)^{R+\nu_1}} \frac{\overset{2R}{*}}{\Gamma(R)} \frac{\Gamma(R+\nu_2)}{(q^2-m_2^2)^{R+\nu_2}} = \left(-\frac{\partial}{\partial q^2}\right)^{R-1} [\nu_1, \nu_2](q^2)$$

$$= \int_0^1 d\zeta \frac{\zeta^{R-1+\nu_1}(1-\zeta)^{R-1+\nu_2}\Gamma(R+\nu_1+\nu_2)}{\left[\zeta(1-\zeta)q^2 - \zeta m_1^2 - (1-\zeta)m_2^2\right]^{R+\nu_1+\nu_2}}$$

$$\frac{2q\,\Gamma(1+R+\nu_1)}{(q^2-m_1^2)^{1+R+\nu_1}} \frac{\overset{2R}{*}}{\Gamma(R)} \frac{\Gamma(R+\nu_2)}{(q^2-m_2^2)^{R+\nu_2}} = 2q\left(-\frac{\partial}{\partial q^2}\right)^{R} [\nu_1, \nu_2](q^2)$$

$$= 2q \int_0^1 d\zeta \frac{\zeta^{R+\nu_1}(1-\zeta)^{R+\nu_2}\Gamma(1+R+\nu_1+\nu_2)}{\left[\zeta(1-\zeta)q^2 - \zeta m_1^2 - (1-\zeta)m_2^2\right]^{R+\nu_1+\nu_2}}$$

$$\frac{2q\,\Gamma(1+R+\nu_1)}{(q^2-m_1^2)^{1+R+\nu_1}} \frac{\overset{2R}{*}}{\Gamma(R)} \frac{2q\,\Gamma(1+R+\nu_2)}{(q^2-m_2^2)^{1+R+\nu_2}} = -2\frac{\partial}{\partial q} \otimes q \left(-\frac{\partial}{\partial q^2}\right)^{R} [\nu_1, \nu_2](q^2)$$

Nontrivial $\mathbf{SO}_0(1, 2R-1)$-properties are effected by energy–momentum derivatives $(\frac{\partial}{\partial q})^L$.

10. INTERACTIONS AND TANGENT STRUCTURE OF SPACE-TIME

From a representation of a real Lie group and its functions, one can obtain the representation of its Lie algebra with the corresponding functions.

Complex functions of a symmetric space G/H (Section 2), in short $\mathcal{G} \subseteq L^\infty(\mathbb{R}^n)$, are representation coefficients of the group G, e.g states of $\mathbf{SO}_0(1,3)$ in the case of hyperbolic positions $\mathcal{Y}^3 \ni \vec{x} \mapsto d(\vec{x}) = e^{-|m|r}$. Their pointwise multiplication property $\mathcal{G} \bullet \mathcal{G} \subseteq \mathcal{G}$ allows—with Fourier transformation to (energy-)momentum Radon measures $\mathcal{M}(\mathbb{R}^n)$, e.g. $\mathbb{R}^3 \ni \vec{q} \mapsto \tilde{d}(\vec{q}) = \frac{|m|}{(\vec{q}^2+m^2)^2}$ — the definition of a convolution algebra $\tilde{\mathcal{G}} * \tilde{\mathcal{G}} \subseteq \tilde{\mathcal{G}}$.

The transition from Lie group representation coefficients to those of the Lie algebra is effected by derivations $(\partial d) \circ \bar{d}$ with respect to Lie algebra parameters, e.g. for time $\mathbf{D}(1)$ by $\frac{de^{imt}}{dt} e^{-imt} = im$ or for rotations $\mathbf{SU}(2)$ by $(\partial^a e^{i|m|\vec{x}\vec{\sigma}}) \circ e^{-i|m|\vec{x}\vec{\sigma}} = i|m|\sigma^b u(\vec{x})^a_b$. The tangent Lie algebra functions from orthogonally symmetric space functions involve the corresponding derivatives with the two-sphere spread. An important and for hyperbolic position characteristic example are the Yukawa potentials. They arise as the orbit of the Coulomb potential $\omega^{-1}(\vec{x}) = \frac{2}{r}$ acted upon with the $\mathbf{SO}_0(1,3)$-states

$$\mathbb{R}^3 \ni \vec{x} \mapsto \vec{\partial} e^{-|m|r} = \frac{\vec{x}}{2} \frac{\partial}{\partial \frac{r^2}{4}} e^{-|m|r}$$

$$-\frac{1}{|m|} \frac{\partial}{\partial \frac{r^2}{4}} e^{-|m|r} = \frac{2e^{-|m|r}}{r} = \frac{2}{r} \bullet e^{-|m|r}$$

Tangent functions for a symmetric space are obtained with inverse derivatives ω^{-1}, familiar as Green functions of differential equations. They give distributions $\tilde{\omega}^{-1}$ of the translation characters, e.g. energy–momentum distributions for spacetime translations. The action of inverse derivative distributions upon the representation coefficients defines the associated tangent module

$$\omega^{-1} : \mathcal{G} \longrightarrow \log \mathcal{G}, \qquad d \longrightarrow \omega = \omega^{-1} \bullet d$$

$$\tilde{\omega}^{-1} : \tilde{\mathcal{G}} \longrightarrow \log \tilde{\mathcal{G}}, \qquad \tilde{d} \longrightarrow \tilde{\omega} = \tilde{\omega}^{-1} * \tilde{d}$$

Tangent functions, $q \mapsto \tilde{\omega}(q)$, are defined with the same integration contour as the representation functions $q \mapsto \tilde{d}(q)$.

The general situation with respect to group functions—always on space (-time) translations $x \in \mathbb{R}^n$ or no (energy-) momenta $q \in \mathbb{R}^n$: The Fourier transformed Radon (energy-)momentum distributions $\mathcal{M}$ are L^∞-functions of space (-time) translations—they are used for group representation coefficients. The Fourier transformed space (-time) function classes L^1 are a dense subspace of the continuous functions $\mathcal{C}_\infty$ of (energy-)momenta which vanish for infinity (Folland, 1995; Treves, 1967)—they are used for Lie algebra representation coefficients.

L^∞ is the dual of L^1. $\mathcal{C}_\infty$ contains the compactly supported functions $\mathcal{C}_c$ whereof the Radon distributions are the dual.

All those spaces with hydrogen ground state and Yukawa potential as an example for representation coefficients are summarized with their Fourier transformation and multiplicative properties—pointwise or convolutive—as follows

	Fourier transformation	(energy-) Momentum functions	Space(-time) functions
For Lie group G	$\mathcal{M} \longrightarrow L^\infty, \tilde{d} \to d$	$\mathcal{M} * \mathcal{M} \subseteq \mathcal{M}$	$L^\infty \bullet^\infty \subseteq L^\infty$
Representation coefficients	$\int \frac{d^3q}{\pi^2} e^{i\vec{q}\vec{x}} \frac{\|m\|}{(\vec{q}^2+m^2)^2} = e^{-\|m\|r}$	$\tilde{d}_1 * \tilde{d}_2(q)$	$d_1 \bullet d_2(x)$
For Lie algebra $\log G$	$\mathcal{C}_\infty \leftarrow L^1, \tilde{\omega} \leftarrow \omega$	$\mathcal{C}_\infty \bullet \mathcal{C}_\infty \subseteq \mathcal{C}_\infty$	$L^1 * L^1 \subseteq L^1$
representation coefficients	$\frac{1}{\vec{q}^2+m^2} = \int \frac{d^3x}{4\pi} e^{-\vec{q}\vec{x}} \frac{e^{-\|m\|r}}{r}$	$\tilde{\omega}_1 \bullet \tilde{\omega}_2(q)$	$\omega_1 * \omega_2(x)$

$$\begin{array}{llll} \text{for Lie group:} & \mathcal{M} \to L^\infty & \mathcal{M} = (\mathcal{C}_c)' & L^\infty = (L^1)' \\ & \cup \quad\ \ \cup \ \text{ with } & \tilde{\omega}^{-1} * \downarrow \qquad \text{and} & \omega^{-1} \bullet \downarrow \\ \text{for Lie algebra:} & L^1 \to \mathcal{C}_\infty & \mathcal{C}_\infty \supset \mathcal{C}_c & L^1 \end{array}$$

For a nonabelian noncompact group, the representation coefficients $\log \mathcal{G}$ of the Lie algebra $\log G$ describe interactions. Their Fourier transforms $\log \tilde{\mathcal{G}}$ constitute a subspace of the (energy-)momentum functions $\mathcal{C}_\infty$

$$\tilde{\mathcal{G}} \subseteq \mathcal{M}, \quad \log \tilde{\mathcal{G}} = \tilde{\omega}^{-1} * \tilde{\mathcal{G}} \subseteq \mathcal{C}_\infty$$

$\log \tilde{\mathcal{G}}$ is a module for the convolutive action with the group representation distributions $\tilde{\mathcal{G}}$

$$\log \tilde{\mathcal{G}} * \tilde{\mathcal{G}} \subseteq \log \tilde{\mathcal{G}} \text{ with } \tilde{\omega}_1 * \tilde{d}_2 = \tilde{\omega}^{-1} * \tilde{d}_1 * \tilde{d}_2 = \tilde{\omega}_{1*2}$$

A requirement of convolutive stability for (energy-)momentum tangent functions themselves or pointwise multiplicative stability for their space (-time)-dependent Fourier transforms does not make sense as seen, e.g., in the pointwise multiplication of the off-shell contributions of Feynman propagators ('divergencies') or in the pointwise square $\frac{e^{-2|m|r}}{r^2}$ of a Yukawa potential as tangent representation coefficient, which is not used as basic potential as tangent representation coefficient, which is not used as basic potential. Its convolution square, however, makes sense as group representation coefficient

$$\mathcal{C}_\infty(\mathbb{R}^3) \ni \frac{|m|}{\vec{q}^2+m^2} \leftrightarrow \frac{e^{-|m|r}}{r} \in L^1(\mathbb{R}^3)$$

$$\frac{1}{\vec{q}^2+m^2} \bullet \frac{1}{\vec{q}^2+m^2} = \frac{1}{(\vec{q}^2+m^2)^2} \leftrightarrow \frac{e^{-|m|r}}{r} * \frac{e^{-|m|r}}{r} = \frac{2\pi}{|m|} e^{-|m|r}$$

10.1. Tangent Modules for Abelian Group

There is not much interaction for abelian groups: With the isomorphy of the abelian group $\mathbf{D}(1)$ (one-dimensional future $\mathbb{R}_+$) and its Lie algebra $\mathbb{R}$ also the representation algebra $\tilde{\mathcal{D}}^1$ is isomorphic to its tangent module $\log \tilde{\mathcal{D}}^1$. It is generated with the inverse derivative $(\frac{d}{dt})^{-1} \sim \frac{1}{q} = \tilde{\omega}^{-1}(q)$

$$\frac{1}{im}\frac{d}{dt}e^{imt} = e^{imt} = \oint \frac{dq}{2i\pi}\frac{1}{q-m}e^{iqt}$$

$$\begin{array}{|c|}\hline \text{time } \frac{1}{q} \overset{1}{*} \tilde{\mathcal{D}}^1 \to \log \tilde{\mathcal{D}}^1 \\ \text{with } \left(\overset{1}{*}, q\right) = \left(\frac{*}{2i\pi}, q - io\right) \\ \hline \frac{1}{q} \overset{1}{*} \frac{1}{q-m} = \frac{1}{q-m} \\ \frac{1}{q-m_1} \overset{1}{*} \frac{1}{q-m_2} = \frac{1}{q-(m_1+m_2)} \\ \hline \end{array}$$

The hyperboloid $\mathcal{Y}^1$ (one-dimensional abelian position) with poles for dual invariants has the inverse derivative distribution $(\frac{d}{dz})^{-1} \sim \frac{q}{q^2} = \tilde{\omega}^{-1}(q)$ with the sign distribution as Fourier transform

$$-\frac{1}{|m|}\frac{\partial}{\partial z}e^{-|mz|} = \epsilon(z)e^{-|mz|} = \int \frac{dq}{\pi}\frac{iq}{q^2+m^2}e^{-iqz}$$

$$\begin{array}{|c|}\hline \text{1-position } \frac{q}{q^2} \overset{1}{*} \tilde{\mathcal{Y}}^1 \to \log \tilde{\mathcal{Y}}^1 \\ \text{with } \left(\overset{1}{*}, q^2\right) = \left(\frac{*}{\pi}, q^2 o^2\right) \\ \hline \frac{q}{q^2} \overset{1}{*} \frac{|m|}{q^2+m^2} = \frac{q}{q^2+m^2} \\ \frac{q}{q^2+m_1^2} \overset{1}{*} \frac{|m|}{q^2+m_2^2} = \frac{q}{q^2+m_+^2} \\ \hline \end{array}$$

The distributions of the abelian sphere $\Omega^1 \cong \mathbf{SO}(2)$ have a different integration contour

$$\pm\frac{1}{i|m|}\frac{\partial}{\partial z}e^{\pm i|mz|} = \epsilon(z)e^{\pm i|mz|} = \pm\int \frac{dq}{\pi}\frac{q}{q^2 \mp io - m^2}e^{-iqz}$$

$$\begin{array}{|c|}\hline \text{circle } \frac{q}{q^2} \overset{1}{*} \tilde{\Omega}^1 \to \log \tilde{\Omega}^1 \\ \text{with } (\overset{1}{*}, q^2) = (\pm\frac{*}{i\pi}, q^2 \mp io) \\ \hline \frac{q}{q^2} \overset{1}{*} \frac{|m|}{q^2-m^2} = \frac{q}{q^2-m^2} \\ \frac{q}{q^2-m_1^2} \overset{1}{*} \frac{|m|}{q^2-m_2^2} = \frac{q}{q^2-m_+^2} \\ \hline \end{array}$$

10.2. Interactions of Hyperbolic Position

The tangent functions (interactions) for hyperbolic position $\mathcal{Y}^3$ with $\vec{\partial} = \frac{\vec{x}}{2}\frac{\partial}{\partial\frac{r^2}{4}}$ are the orbit of the inverse scalar derivative distribution $(\vec{\partial}^2)^{-1} \sim \frac{1}{\vec{q}^2} = \tilde{\omega}^{-1}(q)$ with the Kepler factor (Coulomb and Newton potential) as Fourier transform. They are the Yukawa potentials

$$-\frac{1}{|m|}\frac{\partial}{\partial\frac{r^2}{4}}e^{-|m|r} = 2\frac{e^{-|m|r}}{r} = \int \frac{d^3q}{\pi^2}\frac{1}{\vec{q}^2+m^2}e^{-i\vec{q}\vec{x}}$$

position $\frac{1}{\vec{q}^2} \overset{3}{*} \tilde{\mathcal{Y}}^3 \to \log \tilde{\mathcal{Y}}^3$				
with $\mathbf{SO}(3)$ and $(\overset{3}{*}, \vec{q}^2) = \left(\frac{*}{\pi^2}, \vec{q}^2 + o^2\right)$				
$\frac{1}{\vec{q}^2}$	$\overset{3}{*}$	$\frac{|m|}{(\vec{q}^2+m^2)^2}$	$=$	$\frac{1}{\vec{q}^2+m^2}$
$\frac{1}{\vec{q}^2+m_1^2}$	$\overset{3}{*}$	$\frac{|m_2|}{\left(\vec{q}^2+m_2^2\right)^2}$	$=$	$\frac{1}{\vec{q}^2+m_+^2}$

The characteristic minimal nonabelian case is generalized to the odd-dimensional hyperboloids

$$-\frac{1}{|m|}\frac{\partial}{\partial\frac{r^2}{4}}e^{-|m|r} = 2\frac{e^{-|m|r}}{r} = \int \frac{2d^{2R-1}q}{|\Omega^{2R-1}|}\frac{1}{(\vec{q}^2+m^2)^{R-1}}e^{-i\vec{q}\vec{x}}$$

position $\frac{1}{(\vec{q}^2)^{R-1}} \overset{2R-1}{*} \tilde{\mathcal{Y}}^{2R-1} \to \log \tilde{\mathcal{Y}}^{2R-1}, \quad R = 2, 3, \ldots$				
with $\mathbf{SO}(2R-1)$ and $(\overset{2R-1}{*}, \vec{q}^2) = \left(\frac{*\,2}{|\Omega^{2R-1}|}, \vec{q}^2 + o^2\right)$				
$\frac{1}{(\vec{q}^2)^{R-1}}$	$\overset{2R-1}{*}$	$\frac{|m|}{(\vec{q}^2+m^2)^R}$	$=$	$\frac{1}{(\vec{q}^2+m^2)^{R-1}}$
$\frac{1}{\left(\vec{q}^2+m_1^2\right)^{R-1}}$	$\overset{2R-1}{*}$	$\frac{|m_2|}{\left(\vec{q}^2+m_2^2\right)^R}$	$=$	$\frac{1}{\left(\vec{q}^2+m_+^2\right)^{R-1}}$

and with the real–imaginary transition to the odd-dimensional spheres

$$\mp\frac{1}{i|m|}\frac{\partial}{\partial\frac{r^2}{4}}e^{\pm i|m|r} = 2\frac{e^{\pm i|m|r}}{r} = \pm\int \frac{2d^{2R-1}q}{i|\Omega^{2R-1}|}\frac{1}{(\vec{q}^2 \mp io - m^2)^{R-1}}e^{-i\vec{q}\vec{x}}$$

spheres $\frac{1}{(\vec{q}^2)^{R-1}} \overset{2R-1}{*} \tilde{\Omega}^{2R-1} \to \log \tilde{\Omega}^{2R-1}, \quad R = 2, 3, \ldots$	
with $\mathbf{SO}(2R-1)$ and $(\overset{2R-1}{*}, \vec{q}^2) = \left(\pm \frac{*\,2}{\lvert\Omega^{2R-1}\rvert}, \vec{q}^2 \mp io\right)$	
$\frac{1}{(\vec{q}^2)^{R-1}} \overset{2R-1}{*} \frac{\lvert m\rvert}{(\vec{q}^2+m^2)^R} = \frac{1}{(\vec{q}^2+m^2)^{R-1}}$	
$\frac{1}{\left(\vec{q}^2+m_1^2\right)^{R-1}} \overset{2R-1}{*} \frac{\lvert m_2\rvert}{\left(\vec{q}^2+m_2^2\right)^R} = \frac{1}{\left(\vec{q}^2+m_+^2\right)^{R-1}}$	

10.3. Interactions of Causal Spacetime

The convolutive action of representation coefficients of two-dimensional spacetime on the inverse derivative $\partial^{-1} \sim \frac{q}{q^2} = \tilde{\omega}^{-1}(q)$

$$\int \frac{d^2q}{i\pi} \frac{q}{(q-io)^2} e^{iqx} = \vartheta(x_0)\pi x \delta\left(\frac{x^2}{4}\right), \quad \begin{cases} \int \frac{dz}{2\pi} \int \frac{d^2q}{i\pi} \frac{q}{(q-io)^2} e^{iqx} = 2\vartheta(t) \\ \int \frac{dt}{2\pi} \int \frac{d^2q}{i\pi} \frac{q}{(q-io)^2} e^{iqx} = \epsilon(z) \end{cases}$$

produces pole structures for the spacetime tangent functions with an additional ζ-integration

spacetime $\frac{q}{q^2} \overset{2}{*} \tilde{\mathcal{D}}^2 \to \log \tilde{\mathcal{D}}^2$	
with $\mathbf{SO}_0(1,1)$ and $\left(\overset{2}{*}, q^2\right) = \left(\frac{*}{2i\pi}, (q-io)^2\right)$	
$\frac{q}{q^2} \overset{2}{*} \frac{1}{q^2-m^2} = \int_0^1 d\zeta \frac{q}{\zeta q^2 - m^2}$	
$\frac{q}{q^2-m_1^2} \overset{2}{*} \frac{1}{q^2-m_2^2} = \int_0^1 d\zeta \frac{(1-\zeta)q}{\zeta(1-\zeta)q^2 - \zeta m_1^2 - (1-\zeta)m_2^2}$	

The inverse derivative distribution for general even-dimensional spacetime has the generalized Coulomb force as position projection

$$\int \frac{2d^{2R}q}{i\lvert\Omega^{2R-1}\rvert} \frac{q}{(q-io)^2} e^{iqx} = \vartheta(x_0)\pi x \Gamma(R) \delta^{(R-1)}\left(-\frac{x^2}{4}\right)$$

$$\int \frac{\lvert\Omega^{2R-1}\rvert d^{2R-1}x}{(2\pi)^{2R}} \int \frac{2d^{2R}q}{i\lvert\Omega^{2R-1}\rvert} \frac{q}{(q-io)^2} e^{iqx} = 2\vartheta(t)$$

$$\int \frac{dt}{2\pi} \int \frac{2d^{2R}q}{i\lvert\Omega^{2R-1}\rvert} \frac{q}{(q-io)^2} e^{iqx} = \frac{\vec{x}}{r} \frac{\Gamma(2R-1)}{(r^2)^{R-1}}$$

and the action on the representation coefficients

$$\begin{array}{c}
\text{spacetime} \quad \frac{q}{q^2} \overset{2R}{*} \tilde{\mathcal{D}}^{2R} \to \log \tilde{\mathcal{D}}^{2R}, \quad R = 1, 2, 3 \ldots \\
\text{with} \quad \mathbf{SO}_0(1, 2R-1) \quad \text{and} \quad (\overset{2R}{*}, q^2) = \left(-\frac{*(-1)^R}{i|\Omega^{2R-1}|}, (q-io)^2\right)
\end{array}$$

$$\begin{aligned}
\frac{q}{q^2} \overset{2R}{*} \frac{1}{(q^2-m^2)^R} &= \int_0^1 d\zeta \frac{(1-\zeta)^{R-1} q}{\zeta q^2 - m^2} \\
&= \int_{m^2}^{\infty} \frac{dM^2}{M^2} \left(\frac{M^2-m^2}{M^2}\right)^{R-1} \frac{q}{q^2-M^2} \\
\frac{q}{q^2-m_1^2} \overset{2R}{*} \frac{1}{\left(q^2-m_2^2\right)^R} &= \int_0^1 d\zeta \frac{(1-\zeta)^{R_q}}{\zeta(1-\zeta)q^2 - \zeta m_1^2 - (1-\zeta)m_2^2} \\
\frac{q}{q^2-m_1^2} \overset{2R}{*} \frac{2q\,R}{\left(q^2-m_2^2\right)^{1+R}} &= -\frac{\partial}{\partial q} \otimes q \int_0^1 d\zeta \frac{(1-\zeta)^R}{\zeta(1-\zeta)q^2 - \zeta m_1^2 - (1-\zeta)m_2^2}
\end{aligned}$$

The time projection displays the embedded time translation representations, and the position projection the Yukawa forces

$$\int \frac{2d^{2R}q}{i|\Omega^{2R-1}|} \frac{q}{(q-io)^2 - m^2} e^{iqx} = \pi x \Gamma(R) \left(\frac{\partial}{\partial \frac{x^2}{4}}\right)^R \vartheta(x) \mathcal{J}_0(|mx|)$$

$$\int \frac{|\Omega^{2R-1}| d^{2R-1}x}{(2\pi)^{2R}} \int \frac{2d^{2R}q}{i|\Omega^{2R-1}|} \frac{q}{(q-io)^2 - m^2} e^{iqx} = \vartheta(t) 2 \cos mt$$

$$\int \frac{dt}{2\pi} \int \frac{2d^{2R}q}{i|\Omega^{2R-1}|} \frac{q}{(q-io)^2 - m^2} e^{iqx} = \frac{\vec{x}}{2|m|} \Gamma(R) \left(-\frac{\partial}{\partial \frac{r^2}{4}}\right)^R e^{-i|m|r}$$

11. PROJECTIVE ENERGY–MOMENTA

The (energy-)momentum-dependent tangent functions $\log \tilde{\mathcal{G}} \subseteq \mathcal{C}_\infty$ with space (-time) functions $\log \mathcal{G} \subseteq L^1$ can be multiplied $\log \tilde{\mathcal{G}} \bullet \log \tilde{\mathcal{G}} \subseteq \log \tilde{\mathcal{G}}$ and convoluted $\log \mathcal{G} * \log \mathcal{G} \subseteq \log \mathcal{G}$.

A tangent function $q \mapsto \tilde{\omega}(q)$ is projection valued at q_0 if it has value 1 for this (energy-)momentum, called a projective point of $\tilde{\omega}$

$$\tilde{\omega} \in \log \tilde{\mathcal{G}} \quad \text{projective at} \quad q_0 \Leftrightarrow \tilde{\omega}(q_0) = \mathbf{1}$$

$$\text{with} \quad \tilde{\omega} =^{\oplus} \int d^n q\, \tilde{\omega}(q) |q\rangle\rangle q| \text{ and } \langle q'|q\rangle = \delta(q-q')$$

Projection valued tangent functions will be used to define dual pairs of representation and inverse derivative distributions and for eigenvalue equations of representation invariants.

11.1. Duality of Group and Lie Algebra Representations

The invariants of a group representation coincide with those of the corresponding Lie algebra representation. A pair with a corresponding group and inverse derivative distribution defines a projector.

The representation algebra $\mathcal{G} \subseteq L^\infty = (L^1)'$ is dual to the tangent module $\log \mathcal{G} \subseteq L^1$. The dual product of representation coefficients of the symmeytric space G/H and its tangent space is the convolution at trivial (energy-)momentum $q = 0$

$$\begin{aligned} \log \mathcal{G} \times \mathcal{G} &\longrightarrow \mathbb{C}, \qquad \langle \omega, d \rangle = \textstyle\int d^n x\, \omega(x) d(x) = \\ \log \tilde{\mathcal{G}} \times \tilde{\mathcal{G}} &\longrightarrow \mathbb{C}, \qquad \langle \tilde{\omega}, \tilde{d} \rangle = \textstyle\int d^n q\, \tilde{\omega}(-q) \tilde{d}(x) = \tilde{\omega} * \tilde{d}(0) \end{aligned}$$

A representation distribution is dual to the inverse derivative distribution $\tilde{\omega}^{-1}$ if it has projective point for trivial (energy-)momentum $q = 0$

$$\text{projective at } q_0 = 0 \Leftrightarrow \text{ dual pair } (\tilde{\omega}^{-1}, \tilde{d}) \Leftrightarrow \tilde{\omega}(0) = \tilde{\omega}^{-1} * \tilde{d}(0) = 1$$

A dual pair of representations for time and nonabelian hyperboloids and spheres consists of one basic representation distribution and an inverse derivative distribution with the same continuous real invariant as singularity and normalization respectively.

$$\text{abelian } \mathbf{D}(1) : -\frac{m}{q} \overset{1}{*} \frac{1}{q-m} = -\frac{m}{q-m} \overset{q=0}{=} 1$$

$$\left.\begin{array}{r} \mathcal{Y}^1 \text{ with } \epsilon = +1 \\ \Omega^1 \text{ with } \epsilon = -1 \end{array}\right\} \frac{q\epsilon|m|}{q^2} \overset{1}{*} \frac{q}{q^2 + \epsilon m^2} = \frac{\epsilon m^2}{q^2 + \epsilon m^2} \overset{q=0}{=} 1$$

$$\left.\begin{array}{r} \text{hyperboloids } \mathcal{Y}^{2R-1}, \epsilon = +1 \\ \text{spheres } \Omega^{2R-1}, \epsilon = -1 \\ \text{for} \quad R = 2, 3, \ldots \end{array}\right\} \frac{(\epsilon m^2)^{R-1}}{(\vec{q}^2)^{R-1}} \overset{2R-1}{*} \frac{|m|}{(\vec{q}^2 + \epsilon m^2)R}$$

$$= \frac{(\epsilon m^2)^{R-1}}{(\vec{q}^2 + \epsilon m^2)^{R-1}} \overset{\vec{q}^2=0}{=} 1$$

11.2. The Ratio of the Invariant Masses for Spacetime

In the dual product for homogeneous even-dimensional spacetimes $\mathcal{D}^{2R}$, starting with Cartan spacetime $\mathbf{D}(1) \times \mathbf{SO}_0(1, 1)$ and Minkowski spacetime $\mathbf{D}(2) \cong \mathbf{GL}(\mathbb{C})/\mathbf{U}(2)$, the residual vector spacetime representation distribution involves two continuous real invariants for real rank 2 in contrast to the time and position representations with only one invariant. The spacetime inverse derivative distribution $\frac{q|m_0|}{q^2}$ comes with the intrinsic mass unit m_0^2 of the residual representation of homogeneous spacetime with $R = 1, 2, \ldots$

$$\frac{2d^{2R}q}{\left(q^2 - m_\kappa^2\right)^R} \frac{q|m_0|}{q^2 - m_0^2} \cdot \mapsto \frac{2d^{2R}q}{(q^2 - \kappa^2)^R} \frac{q}{q^2 - 1} \text{ with } \kappa^2 = \frac{m_\kappa^2}{m_0^2}$$

$$\frac{2}{(q^2 - \kappa^2)^R} \frac{q}{q^2 - 1} = \int_{\kappa^2}^1 \frac{d\eta^2}{1 - \eta^2} \left(\frac{1 - \eta^2}{1 - \kappa^2}\right)^R \frac{2q\, R}{(q^2 - \eta^2)^{1+R}}$$
$$= -\frac{\partial}{\partial q} \int_0^1 d\xi \frac{(1 - \xi)^{R-1}}{[q^2 - \xi - (1 - \xi)\kappa^2]^R}$$

The invariant singularities in $q^2 - \xi m_0^2 - (1 - \xi)m_\kappa^2$ are on a finite line with $\xi \in [0, 1]$.

Compatible with the finite integration and in contrast to the energy and momentum pole functions for time and position, the residual product of the even-dimensional spacetime representation and the inverse derivative distribution

$$\frac{q}{q^2} \overset{2R}{*} \frac{2}{(q^2 - \kappa^2)^R} \frac{q}{q^2 - 1} = \frac{\partial}{\partial q} \otimes q\tilde{\omega}_{2R}^0(q^2, \kappa^2)$$

does not produce a rational complex function with a q^2-pole, but a finite integration over the square $(\zeta, \xi) \in [0, 1]^2$ for a pole distribution with $\zeta q^2 - \xi m_0^2 - (1 - \xi)m_\kappa^2$

$$\tilde{\omega}_{2R}^0(q^2, \kappa^2) = -\int_{\kappa^2}^1 \frac{d\eta^2}{1 - \eta^2} \left(\frac{1 - \eta^2}{1 - \kappa^2}\right)^R \int_0^1 d\zeta \frac{(1 - \zeta)^{R-1}}{\zeta q^2 - \eta^2}$$
$$= -\int_0^1 d\xi \int_0^1 d\zeta \frac{(1 - \xi)^{R-1}(1 - \zeta)^{R-1}}{\zeta q^2 - \xi - (1 - \xi)\kappa^2}$$

The inverse derivative distribution and the residual spacetime representation function are dual to each other by fulfilling the condition for the mass ratio κ^2 in the scalar causal measure

$$\tilde{\omega}_{2R}^0(0, \kappa^2) = -\frac{1}{R} \log_R \kappa^2 = 1$$

The R-tails of the logarithm in the $(m_0^2 - m_\kappa^2) \sim (1 - \kappa^2)$-expansion

$$\mathbb{R}_+ \ni \kappa^2 \mapsto \log_R \kappa^2 = -\frac{1}{(1 - \kappa^2)^R} \int_{\kappa^2}^1 d\eta^2 \frac{(1 - \eta^2)^{R-1}}{\eta^2} \text{ for } R = 1, 2, \ldots$$
$$= \frac{1}{(1 - \kappa^2)^R} \left[\log \kappa^2 + \sum_{k=1}^{R-1} \frac{(1 - \kappa^2)^k}{k}\right]$$
$$= -\sum_{k=R}^{\infty} \frac{(1 - \kappa^2)^{k-R}}{k}$$

increase monotonically for $\kappa^2 < 1$

$$\kappa^2 \in (0,1) : \frac{d}{d\kappa^2} \log_R \kappa^2 > 0, \quad \begin{cases} \kappa^2 \ll 1 : \log_R \kappa^2 \sim \log \kappa^2 + \sum_{k=1}^{R-1} \frac{1}{k} \\ \kappa^2 = 1 : \log_R 1 = -\frac{1}{R} \end{cases}$$

For Cartan spacetime without rotation degrees, both invariants coincide

$$R = 1 : -1 = \log_1 \kappa_2 = \frac{\log \kappa^2}{1-\kappa^2} \Rightarrow \kappa^2 = \frac{m_\kappa^2}{m_0^2} = 1$$

For spacetimes $R \geq 2$ with nontrivial rotations, the mass ratio goes with the exponential of the spacetime rank

$$R = 2 : -2 = \frac{\log \kappa^2 + 1 - \kappa^2}{(1-\kappa^2)^2} \Rightarrow \kappa^2 = \frac{m_\kappa^2}{m_0^2} \sim e^{-3} \sim \frac{1}{20.1}$$

$$R = 2, 3, \ldots : -R = \log_R \kappa^2 \Rightarrow \kappa^2 = \frac{m_\kappa^2}{m_0^2} \sim \exp\left[-R - \sum_{k=1}^{R-1} \frac{1}{k}\right] \text{for } \kappa^2 \ll 1$$

12. PRODUCT INVARIANTS AND MASSES

Starting from one defining representation for time, position or spacetime, its convolution products define product representations. The projective points of the tangent (energy-) momenta functions give the translation invariants. To motivate the general concept of a polynomial representation algebra with its associated tangent module and the related eigenvalue equations, these structures are exemplified first for the abelian time translations $\mathbf{D}(1) \cong \mathbb{R}$.

12.1. The Linear Spectrum for Time Translations

The representation energy distribution $\frac{1}{q-m} \in \tilde{\mathcal{D}}^1$ for time translations $\mathbb{R} \ni t \longmapsto e^{imt}$ with a frequency $\mathbb{R} \ni m \neq 0$ generates, by the convolution powers, a polynomial representation algebra

$$\tilde{\mathcal{D}}^1(m) : \{\left(\frac{1}{q-m}\right)^{*k} = \underbrace{\frac{1}{q-m} \overset{1}{*} \cdots \overset{1}{*} \frac{1}{q-m}}_{k \text{ times}} = \frac{1}{q-km} | k = 1, 2, \ldots\}$$

The poles give the invariants for the power representations $t \longmapsto e^{ikmt}$—the equidistant oscillator energies.

The inverse time derivative as derivative distribution is dual to the generating representation

$$-\frac{m}{q} \overset{1}{*} \frac{1}{q-m} = -\frac{m}{q-m} \overset{q=0}{=} 1$$

Its action on the polynomial representation algebra defines the associated tangent module

$$\log \tilde{\mathcal{D}}^1(m): \left\{ -\frac{m}{q} \overset{1}{*} \frac{1}{q-km} = \frac{m}{q-km} | k = 1, 2, \ldots \right\}$$

The eigenvalues as invariants for the product representations are the projective energy points of the tangent functions, i.e. the solutions of the eigenvalue equations

$$k = 1, 2, \ldots : -\frac{m}{q-km} = 1 \Rightarrow q = m_{k-1} = (k-1)m = 0, m, 2m, \ldots$$

12.2. Eigenvalue Equations for Product Invariants

In general, a representation (energy-)momentum distribution $\tilde{d}(m) \in \tilde{\mathcal{G}}$ with invariants m generates, by its convolution powers (involving tensor powers), the associated polynomial representation algebra

$$\tilde{\mathcal{G}}(m): \ \{\tilde{d}^{*k}(m) | \tilde{d}^{*k}(m, q) = \underbrace{\tilde{d}(m) * \cdots * \tilde{d}(m)}_{k \text{ times}}(q), k = 1, 2, \ldots\}$$

The inverse derivative distribution, dual to $\tilde{d}(m)$

$$\tilde{\omega}^{-1}(m) * \tilde{d}(m)(q) \overset{q=0}{=} 1$$

defines the associated tangent module

$$\tilde{\omega}^{-1}(m) * \tilde{\mathcal{G}}(m) = \log \tilde{\mathcal{G}}(m) : \{\tilde{\omega}^{k-1}(m) = \tilde{\omega}^{-1}(m) * \tilde{d}^{*k}(m) | k = 1, 2, \ldots\}$$

The invariants for the power representations are the projective (energy-)momenta of the tangent functions given by the solutions of the eigenvalue equations

$$k = 1, 2, \ldots : \ \tilde{w}^{k-1}(m, q) = \mathbf{1} \Rightarrow q = m_{k-1}$$

12.3. The Quadratic Spectrum for Hyperbolic Position

For position $\mathcal{Y}^3 \cong \mathbf{SO}_0(1, 3)/\mathbf{SO}(3)$, the polynomial representation algebra is visible in the nonrelativistic hydrogen bound states. The Coulomb potential is the dual tangent function for the $\mathbf{SO}_0(1,3)$-state $\vec{x} \longmapsto e^{-|m|r}$

$$d_0(\vec{x}) = e^{-|m|r} = \int \frac{d^3q}{\pi^2} \frac{|m|}{(\vec{q}^2+m^2)^2} e^{-i\vec{q}\vec{x}}$$

$$\omega^{-1}(\vec{x}) = -|m| \frac{\partial}{\partial \frac{r^2}{4}} d_0(\vec{x}) = 2\frac{m^2}{r} = \int \frac{d^3q}{\pi^2} \frac{m^2}{\vec{q}^2} e^{-i\vec{q}\vec{x}}$$

$$\Rightarrow \tilde{\omega}^{-1} \stackrel{3}{*} \tilde{d}_0(q) = \frac{m^2}{\vec{q}^2} \stackrel{3}{*} \frac{|m|}{(\vec{q}^2+m^2)} = \frac{m^2}{\vec{q}^2+m^2} \stackrel{\vec{q}^2=0}{=} 1 = \int_0^\infty r^2 dr \frac{m^2}{r} e^{-|m|r}$$

The Hamiltonian $H = \frac{\vec{q}^2}{2} - \frac{|m|}{r}$ relates to each other the $\mathbf{SO}_0(1,3)$-invariants for position $\mathcal{Y}^3$ and the time translation invariants (binding energies). The time translation action can be written with the convolution product of the inverse derivative distribution and the Fourier transformed state $\tilde{d}_0$

$$\begin{aligned} 2Hd_0(\vec{x}) = 2E_0 d_0(\vec{x}) &\Longleftrightarrow \left(\vec{q}^2 - \frac{|m|}{\vec{q}^2} \stackrel{3}{*}\right) \tilde{d}_0 = 2E_0 \tilde{d}_0 \\ &\Longleftrightarrow \left(\vec{q}^2 - \frac{|m|}{\vec{q}^2} \stackrel{3}{*}\right) \frac{|m|}{(\vec{q}^2+m^2)^2} \\ &= \frac{\vec{q}^2|m|}{(\vec{q}^2-m^2)^2} - \frac{|m|}{(\vec{q}^2+m^2)^2} = 2E_0 \frac{|m|}{(\vec{q}^2+m^2)^2} \\ &\Rightarrow \; 2E_0 = -m^2 \end{aligned}$$

The polynomial representation algebra for general odd-dimensional nonabelian hyperboloids and spheres with the states $\vec{x} \longmapsto (e^{-kr}, e^{\pm ikr})$ and intrinsic unit

$$\tilde{\mathcal{Y}}^{2R-1}, \tilde{\Omega}^{2R-1}: \left\{ \left(\frac{1}{(\vec{q}^2+\epsilon)^R}\right)^{*k} = \frac{k}{(\vec{q}^2+\epsilon k)^R} | k = 1, 2, \ldots \right\}$$

with $\epsilon = \pm 1$ and $R = 2, 3, \ldots$

gives—via the convolutive action—the momentum-dependent tangent functions (from Kepler potential to Yukawa potentials and spherical waves)

$$\log \tilde{\mathcal{Y}}^{2R-1}, \; \log \tilde{\Omega}^{2R-1}: \qquad \left\{ \tfrac{2}{r} \bullet (e^{-kr}, e^{\pm ikr}) = 2\tfrac{(e^{-kr}, e^{\pm ikr})}{r} | k = 1, 2, \ldots \right\}$$

$$\left\{ \tfrac{\epsilon^{R-1}}{(\vec{q}^2)^{R-1}} \stackrel{2R-1}{*} \tfrac{k}{(\vec{q}^2+\epsilon k^2)^{2R-1}} = \tfrac{\epsilon^{R-1}}{(\vec{q}^2+\epsilon k^2)^{R-1}} | k = 1, 2, \ldots \right\}$$

The eigenvalue equations for the projective momenta $\vec{q}$ give the invariants for the representations of the Lorentz group $\mathbf{SO}_0(1, 2R-1)$ as used for $\mathcal{Y}^{2R-1}$ and of the rotation group $\mathbf{SO}(2R)$ as used for Ω^{2R-1}

$$\frac{1}{(\epsilon\vec{q}^2 + k^2)^{R-1}} = 1 \Rightarrow \epsilon\vec{q}^2 = -k^2 + 1 = -4J(1+J) = 0, -3, -8$$
$$\text{for } k = 1 + 2J = 1, 2, 3, \ldots$$

Through the Hamiltonian, the $\mathbf{SO}_0(1,3)$-invariants can be transmuted into energy eigenvalues

$$2E_J = -\frac{m^2}{k^2}, \quad k = 1 + 2J = 1, 2, \ldots$$

12.4. Invariants of Spacetime Translations—The Mass Zero Solution

The residual representation of $2R$-dimensional spacetime $\mathcal{D}^{2R}$, $R = 1, 2, \ldots$ with the $\mathbf{D}(1)$-pole generates, by its convolution powers, the representation algebra

$$\tilde{\mathcal{D}}^{2R}(\kappa^2) : \left\{ \left(\kappa^{2R}(q) \frac{q}{q^2 - 1} \right)^{*k} | k = 1, 2, \ldots \right\} \text{ with } \kappa^{2R}(q) = \frac{2}{(q^2 - \kappa^2)^R}$$

The mass ratio $\kappa^2 = \frac{m_\kappa^2}{m_0^2}$ is determined by $\log_R \kappa^2 = -R$ from duality with the spacetime translation distribution $\frac{q}{q^2}$.

The convolution powers of the residual spacetime representation have as associated energy–momentum tangent functions

$$\log \tilde{\mathcal{D}}^{2R}(\kappa^2) : \left\{ \tilde{\omega}_{2R}^{k-1}(q^2, \kappa^2) = \frac{q}{q^2} \overset{2R}{*} \left(\kappa^{2R}(q) \frac{q}{q^2 - 1} \right)^{*k} | k = 0, 1, 2, \ldots \right\}$$

There arises the nonabelian causal convolution for even-dimensional spacetime where the full energy–momentum-dependent scalar causal measure is included

$$\tilde{\omega}_{2R}^{-1}(q^2, \kappa^2) = \frac{q}{q^2}$$
$$\tilde{\omega}_{2R}^{-1}(q^2, \kappa^2) = \frac{q}{q^2} \overset{2R}{*}_\kappa \frac{q}{q^2 - 1} \overset{2R}{*}_\kappa \cdots \overset{2R}{*}_\kappa \frac{q}{q^2 - 1}$$
$$\text{with } \overset{2R}{*}_\kappa \cong \frac{i}{|\Omega^{2R-1}|} \delta(q_1 + q_2 - q) \frac{2}{\left(-q_2^2 + \kappa^2\right)^R}$$

The solutions of the eigenvalue equations

$$k = 0, 1, 2, \ldots : \quad \tilde{\omega}_{2R}^{-1}(q^2, \kappa^2) = \mathbf{1}$$

give invariants of tangent spacetime translations. For a positive residual normalization (later) they describe particle masses in Poincaré group representations.

The simplest nontrivial eigenvalue equation for $k = 1$ is decomposable with the two nondecomposable projectors $\{\mathbf{1}_{2R} - \frac{q\otimes q}{q^2}, \frac{q\otimes q}{q^2}\}$

$$\frac{q}{q^2} \overset{2R}{*_\kappa} \frac{q}{q^2-1} = \frac{q}{q^2} \overset{2R}{*} \frac{2}{(q^2-\kappa^2)^R} \frac{q}{q^2-1} = \left[\mathbf{1}_{2R} + q\otimes q \, 2\frac{\partial}{\partial q^2}\right] \tilde{\omega}^0_{2R}(q^2,\kappa^2)$$
$$= \left(\mathbf{1}_{2R} - \frac{q\otimes q}{q^2}\right) \tilde{\omega}^0_{2R}(q^2,\kappa^2) + \frac{q\otimes q}{q^2}\left(1 + 2q^2\frac{\partial}{\partial q^2}\right) \tilde{\omega}^0_{2R}(q^2,\kappa^2)$$

e.g. for the rotation-free case

$$R = 1 : \tilde{\omega}^0_2(q^2,\kappa^2) = -\int_{\kappa^2}^1 \frac{d\eta^2}{1-\kappa^2} \int_0^1 \frac{d\zeta}{\zeta q^2 - \eta^2} \overset{\kappa^2=1}{=} -\int_0^1 \frac{d\zeta}{\zeta q^2 - 1}$$
$$= -\frac{\log(1-q^2)}{q^2}$$

The equation

$$\text{for } q^2 = 0 : \tilde{\omega}^0_{2R}(q^2,\kappa^2) = \mathbf{1}_{2R}$$

has been used earlier as duality condition to determine the ratio κ^2 of the spacetime invariants. It is also interpretable as eigenvalue equation having as solution a trivial invariant $q^2 = m^2 = 0$. The eigenvalue $m^2 = 0$ for the vector field in Minkowski spacetime will be related to massless vector fields with their residual normalization (later) as gauge coupling constant.

13. NORMALIZATION OF TRANSLATION REPRESENTATIONS

Starting from a generating representation, the residue of a product representation defines its normalization. For spacetime, the determination of the residues requires the transition from inverse derivative energy–momentum distributions to the associated distributions for the representations of the spacetime translations.

The exponential from the Lie algebra $\mathbb{R}$ (time translations) to the group $\exp\mathbb{R} = \mathbf{D}(1)$ can be reformulated in the language of residual representations with energy functions by a geometric series

$$e^{imt} = \oint \frac{dq}{2i\pi} \frac{1}{q-m} e^{iqt}$$
$$= \sum_{k=0}^{\infty} \frac{(imt)^k}{k!} = \oint \frac{dq}{2i\pi} \frac{1}{q} \sum_{k=0}^{\infty} \left(\frac{m}{q}\right)^k e^{iqt}$$

$-\frac{m}{q}$ is the inverse derivative energy function for the representation function $\frac{1}{q-m}$.

13.1. Geometric Transformation and Mittag–Leffler Sum

Translation representations are characterized by (energy-) momentum distributions with simple poles. Meromorphic complex functions have only pole singularities. In the compactified complex plane $\bar{\mathbb{C}}$, they constitute the field of rational functions. The representation distributions for one dimension (pole functions) have negative degree

$$\bar{\mathbb{C}} \ni q \longmapsto \rho(q) = \frac{P^n(q)}{P^k(q)} = \frac{\alpha_0 + \alpha_1 q + \cdots + \alpha_n q^n}{\gamma_0 + \gamma_1 q + \cdots + \gamma_k q^k} \in \bar{\mathbb{C}},$$
$$\alpha_j,\ \gamma_j \in \mathbb{C},\ \gamma_k \neq 0, \quad k \leq n$$

The geometric transformations for $\mathbf{D}(1)$ (time) with $z = \frac{q}{m}$

$$z \longmapsto \frac{1}{z} = \tilde{\omega}(z) \longmapsto \frac{\tilde{\omega}(z)}{1 - \tilde{\omega}(z)} = \frac{1}{z-1}$$

are elements of the broken rational (conformal) bijective transformations of the closed complex plane

$$\bar{\mathbb{C}} \ni z \longmapsto \frac{\alpha z + \beta}{\gamma z + \delta} \in \bar{\mathbb{C}}$$

with real coefficients as group isomorphic to

$$g = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathbf{SL}(\mathbb{R}^2) \cong \mathbf{SU}(1,1) \sim \mathbf{SO}_0(1,2)$$

For $\det g = 1$ upper and lower half plane $x \pm io$ remain stable. The eigenvalue $\tilde{\omega}(z_0) = 1$ becomes a pole

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} : \tilde{\omega} \longmapsto \frac{\tilde{\omega}}{1 - \tilde{\omega}},\ (1,0) \longmapsto (\infty, 0)$$

With one fixpoint $\tilde{\omega} = 0$ the transformation is parabolic, i.e. an element of the $\mathbb{R}$-isomorphic subgroup $\begin{pmatrix} 1 & 0 \\ \gamma & 1 \end{pmatrix}$.

The geometric transformation will be generalized in order to associate functions with pole singularities to the complex eigenvalue functions $\tilde{\omega}(z)$ for spacetime

$$z \longmapsto \tilde{\omega}(z) \longmapsto \frac{\tilde{\omega}(z)}{1 - \tilde{\omega}(z)}$$

An eigenvalue $z_0 \in \{z | \tilde{\omega}(z) = 1\}$ gives a pole. If the zero z_0 is simple with $\tilde{\omega}$ holomorphic there, it defines, by geometric transformation of its Taylor series, a

Laurent series (Behnke, 1962) and a residue

$$\begin{aligned}
\tilde{\omega}(z) &= 1 + (z - z_0)\tilde{\omega}'(z_0) + \sum_{k=2}^{\infty} \frac{(z - z_0)^k}{k!} \tilde{\omega}^{(k)}(z_0) \\
\frac{\tilde{\omega}(z)}{1 - \tilde{\omega}(z)} &= \frac{\text{res}(z_0)}{z - z_0} + \sum_{k=0}^{\infty} (z - z_0)^k a_k(z_0) \\
\text{res}(z_0) &= -\frac{1}{\tilde{\omega}'(z_0)}
\end{aligned}$$

Each eigenvalue $\{z_k | \tilde{\omega}(z_k) = 1\}$ has its own principal part. Their sum, called Mittag-Leffler sum, replaces the simple pole for $\mathbf{D}(1)$

$$\begin{aligned}
z \longmapsto \tilde{\omega(z)} \longmapsto \tfrac{\tilde{\omega}(z)}{1 - \tilde{\omega}(z)} &= \sum_{z_k} \tfrac{\text{res}(z_k)}{z - z_k} + \cdots \\
\text{generalizing } z \longmapsto \tfrac{1}{z} \longmapsto \tfrac{\frac{1}{z}}{1 - \frac{1}{z}} &= \tfrac{1}{z - 1}
\end{aligned}$$

Therewith one obtains for an eigenvalue function for spacetime $\mathcal{D}^{2R}$ and its projectors at the invariant solutions

$$\tilde{\omega}(q^2) = \tilde{\omega}^{-1} * \tilde{d}(q^2) = \mathbf{1} \Rightarrow q^2 \in \{m^2\}$$

the transition to complex representation functions $\tilde{\mathcal{G}}_0$ assumed with simple poles

$$\tilde{\mathcal{G}} \longrightarrow \log \tilde{\mathcal{G}} \longrightarrow \tilde{\mathcal{G}}_0, \quad \tilde{d} \longmapsto \tilde{\omega}(q^2) \longmapsto \frac{\tilde{\omega}(q^2)}{\mathbf{1} - \tilde{\omega}(q^2)} = \sum_{m^2} \frac{\text{res}(m^2)}{q^2 - m^2} + \cdots$$

The residue is the negative inverse of the derivative of the energy–momentum tangent function at the invariant

$$\tilde{\omega}(q^2) = \mathbf{1} + (q^2 - m^2) \frac{\partial \tilde{\omega}}{\partial q^2}(m^2) + \cdots \Rightarrow \text{res}(m^2) = -\frac{1}{\frac{\partial \tilde{\omega}}{\partial q^2}(m^2)}$$

A simple pole with positive normalization can be used for the representation of the Poincaré group $\mathbf{SO}_0(1, 2R - 1) \vec{\times} \mathbb{R}^{2R}$. The residue gives the normalization of the associated representation

$$\text{on-shell part } \frac{i}{\pi} \frac{\text{res}(m^2)}{q^2 + io - m^2} = \text{res}(m^2) \delta(q^2 - m^2)$$

13.2. Gauge Coupling Constants as Residues at Mass Zero

In the residual product of the spacetime representation with the dual inverse derivative

$$\frac{q}{q^2} \overset{2R}{*}_\kappa \frac{q}{q^2-1} = \left[\mathbf{1}_{2R} + q \otimes q 2\frac{\partial}{\partial q^2}\right] \tilde{\omega}^0_{2R}(q^2, \kappa^2)$$

$$\frac{\partial \tilde{\omega}^0_{2R}}{\partial q^2}(q^2, \kappa^2) = \int_{\kappa^2}^1 \frac{d\eta^2}{1-\eta^2}\left(\frac{1-\eta^2}{1-\kappa^2}\right)^R \int_0^1 d\zeta \frac{\zeta(1-\zeta)^{R-1}}{(\zeta q^2 - \eta^2)^2}$$

the residual normalization res$(0, \kappa^2)$ for the massless solution $\tilde{\omega}^0_{2R}(0, \kappa^2) = 1$ is given by the inverse of the negative derivative of the eigenvalue function there

$$\begin{aligned} -\frac{1}{\text{res}(0,\kappa^2)} = \frac{\partial \tilde{\omega}^0_{2R}}{\partial q^2}(0, \kappa^2) &= \frac{1}{R(R+1)(1-\kappa^2)^R} \int_{\kappa^2}^1 d\eta^2 \frac{(1-\eta^2)^{R-1}}{\eta^4} \\ &= \frac{1}{R(R+1)}\left[\frac{1}{k^2} + (R-1)\log_R \kappa^2\right] \\ &= \frac{1-R(R-1)_{\kappa^2}}{R(R+1)_{\kappa^2}} \quad \text{with} \quad -\frac{1}{R}\log_R \kappa^2 = 1 \end{aligned}$$

With a small mass ratio

$$\text{for } \kappa^2 \ll 1 : -\text{res}(0, \kappa^2) \sim R(R+1)\kappa^2 \sim R(R+1)\exp\left[-R - \sum_{k=1}^{R-1}\frac{1}{k}\right]$$

one has the numerical values for Cartan and Minkowski spacetime

$$-\text{res}(0,\kappa^2) \sim R(R+1)\frac{m_\kappa^2}{m_0^2} = \begin{cases} 2\frac{m_\kappa^2}{m_0^2} = 2, & R = 1 \\ 6\frac{m_\kappa^2}{m_0^2 - 2m_\kappa^2} \sim \frac{6}{e^3-2} \sim \frac{1}{3}, & R = 2 \end{cases}$$

With the geometric transformation, the Laurent series gives an energy–momentum distribution for a spacetime translation representation with invariant zero and residual normalization. With appropriate integration contour, it can be used as propagator for a mass zero spacetime vector field with coupling constant $-\text{res}(0, \kappa^2)$

$$\mathbf{SO}_0(1, 2R-1)\vec{\times}\mathbb{R}^{2R} : \text{on-shell part } \frac{i}{\pi}\frac{\eta_{jk}\text{res}\,(0,\kappa^2)}{q^2 + io} = \eta_{jk}\text{ res }(0,\kappa^2)\delta(q^2)$$

This vector field has, in addition to an $\mathbf{SO}_0(1, 1)$-related pair with neutral signature, $2R-2$ particle interpretable degrees of freedom which are related to the spherical degrees of freedom $\Omega^{2R-2} \subset \mathcal{D}^{2R}$ and the compact fixgroup $\mathbf{SO}(2R-2)$ in the massless particle fixgroup $\mathbf{SO}(2R-2)\vec{\times}\mathbb{R}^{2R-2}$. Those degrees of freedom have a positive scalar product

$$-\eta_{jk} = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_{2R-1} \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{1}_{2R-2} & 0 \\ 1 & 0 & 0 \end{pmatrix} & \text{for } \mathbf{SO}_0(1, 2R-1)\vec{\times}\mathbb{R}^{2R} \\ \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \cong \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{for } \mathbf{SO}_0(1, 1)\vec{\times}\mathbb{R}^{2} \\ \begin{pmatrix} -1 & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix} \cong \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathbf{1}_2 & 0 \\ 1 & 0 & 0 \end{pmatrix} & \text{for } \mathbf{SO}_0(1, 3)\vec{\times}\mathbb{R}^{4} \end{cases}$$

The particle interpretable degrees of freedom start with Minkowski spacetime $2R = 4$. There, the 2 degrees of freedom with a positive scalar product have left and right polarization for the axial $\mathbf{SO}(2)$-rotations.

If adjoint representations of compact internal degrees of freedom, e.g. of $\mathbf{U}(2)$ hypercharge–isospin, are included, the accordingly normalized residues of the arising mass zero solutions in four-dimensional spacetime may be compared with the coupling constants in the propagators of a massless gauge fields in the standard model of electroweak interactions

$$\mathbf{SO}_0(1, 3)\vec{\times}\mathbb{R}^4 : \frac{-\eta^{jk}G^2}{q^2 + io} \text{ with } G^2 \sim \left(g_1^2, g_2^2|g^2, \gamma^2\right), g_1 g_2 \sim \frac{1}{4.6}$$

Without the introduction of the internal degrees of freedom (Saller, 1998b), only the order of magnitude of the normalizations G^2 can be compared with the residues mentioned earlier for the simple massless poles from tangent representations of spacetime $\mathbf{D}(2) \cong \mathbf{GL}(\mathbb{C}^2)/\mathbf{U}(2)$

$$\text{for } \log \mathbf{D}(2) \cong \mathbb{R}^4 : \quad G^2 \leftrightarrow -\text{res }(0, \kappa^2) \sim \frac{1}{3}.$$

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